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THE IMPLEMENTATION OF A ROUGH SET OF PROJECTIVE MODULE

Gusti Ayu Dwi Yanti[1], Fitriani[2*], Ahmad Faisol[3]

[1,2,3]Department of Mathematics, University of Lampung
Prof. Sumantri Brodjonegoro Street, Bandar Lampung, 35145, Indonesia

*Corresponding author's e-mail: * **fitriani.1984@fmipa.unila.ac.id***

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ABSTRACT

In ring and module theory, one concept is the projective module. A module is said to be projective if it is a direct sum of independent modules. (U, R) *is an approximation space with non-empty set* U *and equivalence relation* R. *If* $X \subseteq U$*, we can form upper approximation and lower approximation.* X *is rough set if* $\overline{Apr}(X) \neq \underline{Apr}(X)$*. The rough set theory applies to algebraic structures, including groups, rings, modules, and module homomorphisms. In this study, we will investigate the properties of the rough projective module.*

Keywords:

Approximation space;
Projective module;
Rough projective module.

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1. INTRODUCTION

The rough set theory was first introduced by a mathematician named Pawlak in 1982 [1]. Many kinds of research on the application of rough set theory have been discussed, one of them being algebraic structures. Biswas and Nanda introduce the rough group and rough subgroups [2]. In 2005, Miao et al. researched rough group, rough subgroup, and their properties [3]. Furthermore, Davvaz and Mahdavipour investigate roughness in Modules [4]. In 2009, the concept of rough prime ideals and rough fuzzy prime ideals in gamma-semigroup was introduced [5]. Furthermore, research was also investigated by Sinha and Prakash in 2014 about rough projective modules [6], and Setyaningsih et al. presented a sub-exact sequence of rough groups [7]. Homomorphism and isomorphism of the rough group were studied in [8]. Nugroho et al. investigate the implementation of a rough set on a group structure [9]. In the same year, Hafifulloh et al. Hafifulloh et al. improved the properties of rough coexact sequences in rough groups [10].

A projective module is one of the most fundamental concepts in algebraic structures [11]. Self-projective modules with π −injective factor modules were studied in [12]. In 2005, Puninski and Rothmaler studied pure-projective modules [13]. A module is projective if it is a direct sum of free modules. Furthermore, Fitriani introduces the relationship between the supplement of a module and the existence of the projective envelope of the factor module in category $\sigma[M]$ [14], Schanuel's lemma in P-poor modules [15].

Based on the definition of rough set and projective modules, we introduce the rough projective module. This research applies a rough set theory to construct a rough projective module on an approximation space. Moreover, this research discusses the properties of the rough projective module.

2. RESEARCH METHODS

The research method depends on upper and lower approximations, rough group, rough ring, rough module, rough module homomorphism, and rough sequence exact. First, the rough set is defined by binary operations and determines the ring rough. Next, a rough module is formed. Mapping the surjective rough module homomorphism can form a rough exact sequence. Then from the rough exact sequence, there is a rough projective module. We use the properties of the rough group and the finite set to construct the rough projective module. Finally, we investigate the properties of the rough projective module.

We give the stages of the research as follows.

1. We define the rough projective module over a rough ring.
2. We analyze the properties of the rough projective module.
3. We construct a rough ring, rough module, rough module homomorphism, rough sequence exact, and rough projective module by using some finite sets.

3. RESULTS AND DISCUSSION

We construct the rough projective module, motivated by definition the rough projective module.

3.1. Rough Projective Module over Rough Ring

Before we define the properties of a rough projective module of a rough ring, we must define rough projective module.

Definition 1. A rough module $\overline{Apr}(P)$ over rough ring $\overline{Apr}(R)$ is rough projective module if and only if every rough exact sequence of

$$0 \to \overline{Apr}(M') \to \overline{Apr}(M) \xrightarrow{p} \overline{Apr}(P) \to 0 \qquad (1)$$

is split exact sequence [6].

Next, we give the construction of a rough projective module in an approximation space.

Example 1. Let $U = \mathbb{Z}_{80} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}, \dots, \overline{75}, \overline{76}, \overline{77}, \overline{78}, \overline{79}\}$, we define $x\theta y$ if only if $x - y = 4a$ for some $a \in \mathbb{Z}$. We have four equivalence classes in the following table based on the equivalence relation.

Table 1. The Equivalence Classes of $\mathbb{Z}_{80}$

The Equivalence Classes	The Elements of the Equivalence Classes
E_1	$\{\overline{0},\overline{4},\overline{8},\overline{12},\overline{16},\overline{20},\overline{24},\overline{28},\overline{32},\overline{36},\overline{40},\overline{44},\overline{48},\overline{52},\overline{56},\overline{60},\overline{64},\overline{68},\overline{72},\overline{76}\}$
E_2	$\{\overline{1},\overline{5},\overline{9},\overline{13},\overline{17},\overline{21},\overline{25},\overline{29},\overline{33},\overline{37},\overline{41},\overline{45},\overline{49},\overline{53},\overline{57},\overline{61},\overline{65},\overline{69},\overline{73},\overline{77}\}$
E_3	$\{\overline{2},\overline{6},\overline{10},\overline{14},\overline{18},\overline{22},\overline{26},\overline{30},\overline{34},\overline{38},\overline{42},\overline{46},\overline{50},\overline{54},\overline{58},\overline{62},\overline{66},\overline{70},\overline{74},\overline{78}\}$
E_4	$\{\overline{3},\overline{7},\overline{11},\overline{15},\overline{19},\overline{23},\overline{27},\overline{31},\overline{35},\overline{39},\overline{43},\overline{47},\overline{51},\overline{55},\overline{59},\overline{63},\overline{67},\overline{71},\overline{75},\overline{79}\}$

Furthermore, we give a rough ring to form rough projective module over rough ring.
Let $R = \{\overline{0},\overline{5},\overline{15},\overline{20},\overline{30},\overline{35},\overline{45},\overline{50},\overline{55},\overline{60},\overline{65},\overline{70},\overline{75}\}$, $\overline{Apr}(R) = E_1 \cup E_2 \cup E_3 \cup E_4 = \mathbb{Z}_{80}$, $\underline{Apr}(R) = \emptyset$.
Next, we define the binary operation $\langle R, +_{80}, \cdot_{80}\rangle$. We will show that R is a rough ring.

s

$+_{80}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{30}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$
$\overline{0}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{30}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$
$\overline{5}$	$\overline{5}$	$\overline{10}$	$\overline{20}$	$\overline{25}$	$\overline{35}$	$\overline{40}$	$\overline{50}$	$\overline{55}$	$\overline{65}$	$\overline{70}$	$\overline{0}$
$\overline{15}$	$\overline{15}$	$\overline{20}$	$\overline{30}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$	$\overline{0}$	$\overline{10}$
$\overline{20}$	$\overline{20}$	$\overline{25}$	$\overline{35}$	$\overline{40}$	$\overline{50}$	$\overline{55}$	$\overline{65}$	$\overline{70}$	$\overline{0}$	$\overline{5}$	$\overline{15}$
$\overline{30}$	$\overline{30}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$	$\overline{0}$	$\overline{10}$	$\overline{15}$	$\overline{25}$
$\overline{35}$	$\overline{35}$	$\overline{40}$	$\overline{50}$	$\overline{55}$	$\overline{65}$	$\overline{70}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{30}$
$\overline{45}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$	$\overline{0}$	$\overline{10}$	$\overline{15}$	$\overline{25}$	$\overline{30}$	$\overline{40}$
$\overline{50}$	$\overline{50}$	$\overline{55}$	$\overline{65}$	$\overline{70}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{30}$	$\overline{35}$	$\overline{45}$
$\overline{60}$	$\overline{60}$	$\overline{65}$	$\overline{75}$	$\overline{0}$	$\overline{10}$	$\overline{15}$	$\overline{25}$	$\overline{30}$	$\overline{40}$	$\overline{45}$	$\overline{55}$
$\overline{65}$	$\overline{65}$	$\overline{70}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{30}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$
$\overline{75}$	$\overline{75}$	$\overline{0}$	$\overline{10}$	$\overline{15}$	$\overline{25}$	$\overline{30}$	$\overline{40}$	$\overline{45}$	$\overline{55}$	$\overline{60}$	$\overline{70}$

1. Based on **Table 2**, $x +_{80} y \in \overline{Apr}(R)$, for every $x, y \in R$
2. Commutative property holds binary operation $+_{80}$ in R.
3. Associative property holds binary operation $+_{80}$in $\overline{Apr}(R)$.
4. There exist $0 \in \overline{Apr}(R)$ such that for every $x \in R$, $x +_{80} 0 = 0 +_{80} x = x$.
5. Every element of R has a rough inverse in R in the following **Table 3**.

Table 3. Invers Table on R

$x \in R$	Inverse of x
$\overline{0}$	$\overline{0}$
$\overline{5}$	$\overline{75}$
$\overline{15}$	$\overline{65}$
$\overline{20}$	$\overline{60}$
$\overline{30}$	$\overline{50}$
$\overline{35}$	$\overline{45}$
$\overline{45}$	$\overline{35}$
$\overline{50}$	$\overline{30}$
$\overline{60}$	$\overline{20}$
$\overline{65}$	$\overline{15}$
$\overline{75}$	$\overline{5}$

6. For every $x, y \in R$ then $x \cdot_{80} y \in \overline{Apr}(R)$. We can see this property in **Table 4**.

Table 4. Table Cayley $\cdot_{80}$ on R

$\cdot_{80}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{30}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{5}$	$\overline{0}$	$\overline{25}$	$\overline{75}$	$\overline{20}$	$\overline{70}$	$\overline{15}$	$\overline{65}$	$\overline{10}$	$\overline{60}$	$\overline{5}$	$\overline{55}$
$\overline{15}$	$\overline{0}$	$\overline{75}$	$\overline{65}$	$\overline{60}$	$\overline{50}$	$\overline{45}$	$\overline{35}$	$\overline{30}$	$\overline{20}$	$\overline{15}$	$\overline{5}$
$\overline{20}$	$\overline{0}$	$\overline{20}$	$\overline{60}$	$\overline{0}$	$\overline{40}$	$\overline{60}$	$\overline{20}$	$\overline{40}$	$\overline{0}$	$\overline{20}$	$\overline{60}$

$\cdot_{80}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{30}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$
$\overline{30}$	$\overline{0}$	$\overline{70}$	$\overline{50}$	$\overline{40}$	$\overline{20}$	$\overline{10}$	$\overline{70}$	$\overline{60}$	$\overline{40}$	$\overline{70}$	$\overline{10}$
$\overline{35}$	$\overline{0}$	$\overline{15}$	$\overline{45}$	$\overline{60}$	$\overline{10}$	$\overline{25}$	$\overline{55}$	$\overline{70}$	$\overline{20}$	$\overline{35}$	$\overline{65}$
$\overline{45}$	$\overline{0}$	$\overline{65}$	$\overline{35}$	$\overline{20}$	$\overline{70}$	$\overline{55}$	$\overline{25}$	$\overline{10}$	$\overline{60}$	$\overline{45}$	$\overline{15}$
$\overline{50}$	$\overline{0}$	$\overline{10}$	$\overline{30}$	$\overline{40}$	$\overline{60}$	$\overline{70}$	$\overline{10}$	$\overline{20}$	$\overline{40}$	$\overline{50}$	$\overline{70}$
$\overline{60}$	$\overline{0}$	$\overline{60}$	$\overline{20}$	$\overline{0}$	$\overline{40}$	$\overline{20}$	$\overline{60}$	$\overline{40}$	$\overline{0}$	$\overline{60}$	$\overline{20}$
$\overline{65}$	$\overline{0}$	$\overline{5}$	$\overline{15}$	$\overline{20}$	$\overline{70}$	$\overline{35}$	$\overline{45}$	$\overline{50}$	$\overline{60}$	$\overline{65}$	$\overline{75}$
$\overline{75}$	$\overline{0}$	$\overline{55}$	$\overline{5}$	$\overline{60}$	$\overline{10}$	$\overline{65}$	$\overline{15}$	$\overline{70}$	$\overline{20}$	$\overline{75}$	$\overline{25}$

7. Associative property holds binary operation $\cdot_{80}$ in $\overline{Apr}(R)$.
8. Left distributive law and right distributive law property holds binary operation $\cdot_{80}$ in $\overline{Apr}(R)$.

So, R is a rough ring in approximation space (U, θ).

Furthermore, we give non-empty set $S, T, V \subseteq U$ to form the rough exact sequence.

Example 2. Let $S = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{65}, \overline{71}, \overline{74}, \overline{77}\}$, then $\overline{Apr}(S) = \mathbb{Z}_{80} = U$.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(S)$ then $a \cdot_{80} (x +_{80} y) = (a \cdot_{80} x) +_{80} (a \cdot_{80} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(S)$ then $(a +_{80} b) \cdot_{80} x = (a \cdot_{80} x) +_{80} (b \cdot_{80} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(S)$ then $(a \cdot_{80} b) \cdot_{80} x = a \cdot_{80} (b \cdot_{80} x)$ and $(a \cdot_{80} b) \cdot_{80} x = b \cdot_{80} (a \cdot_{80} x)$.
4. Given $\overline{1} \in \overline{Apr}(R)$ and $x \in \overline{Apr}(S)$ then $\overline{1} \cdot_{80} x = x$.

Hence, S is a rough module. Now, let $T = \{\overline{0}, \overline{9}, \overline{18}, \overline{27}, \overline{53}, \overline{62}, \overline{71}\}$, then $\overline{Apr}(T) = \mathbb{Z}_{80} = U$. We will show that T is a rough module.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(T)$ then $a \cdot_{80} (x +_{80} y) = (a \cdot_{80} x) +_{80} (a \cdot_{80} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(T)$ then $(a +_{80} b) \cdot_{80} x = (a \cdot_{80} x) +_{80} (b \cdot_{80} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(T)$ then $(a \cdot_{80} b) \cdot_{80} x = a \cdot_{80} (b \cdot_{80} x)$ and $(a \cdot_{80} b) \cdot_{80} x = b \cdot_{80} (a \cdot_{80} x)$.
4. Given $\overline{1} \in \overline{Apr}(R)$ and $x \in \overline{Apr}(T)$ then $\overline{1} \cdot_{80} x = x$.

So, T is a rough module. Furthermore, let $V = \{\overline{0}, \overline{15}, \overline{20}, \overline{30}, \overline{50}, \overline{60}, \overline{65}\}$, then $\overline{Apr}(V) = \mathbb{Z}_{80} = U$. We will show that V is a rough module.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(V)$ then $a \cdot_{80} (x +_{80} y) = (a \cdot_{80} x) +_{80} (a \cdot_{80} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(V)$ then $(a +_{80} b) \cdot_{80} x = (a \cdot_{80} x) +_{80} (b \cdot_{80} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(V)$ then $(a \cdot_{80} b) \cdot_{80} x = a \cdot_{80} (b \cdot_{80} x)$ and $(a \cdot_{80} b) \cdot_{80} x = b \cdot_{80} (a \cdot_{80} x)$.
4. Given $\overline{1} \in \overline{Apr}(R)$ and $x \in \overline{Apr}(V)$ then $\overline{1} \cdot_{80} x = x$.

This result shows that V is a rough module.

The homomorphisms p and v are surjective rough module homomorphism define by $p: \overline{Apr}(T) \to \overline{Apr}(V)$ with $p(a) = a$ mod 80 for every $a \in \overline{Apr}(T)$, $v: \overline{Apr}(V) \to \overline{Apr}(T)$ with $v(a) =$ identity function for every $a \in \overline{Apr}(V)$. Given sequence $0 \to \overline{Apr}(S) \to \overline{Apr}(T) \xrightarrow{p} \overline{Apr}(V) \to 0$. We know that p and v is surjective rough module homomorphism. Therefore, $p \circ v = id_p$ and id_p is identity homomorphism. So, $0 \to \overline{Apr}(S) \to \overline{Apr}(T) \xrightarrow{p} \overline{Apr}(V) \to 0$ is split exact sequence. Next, we will show that rough module $\overline{Apr}(V)$ is rough projective module. We know exact sequence $0 \to \overline{Apr}(S) \to \overline{Apr}(T) \xrightarrow{p} \overline{Apr}(V) \to 0$ is split, then

$$\begin{array}{ccc} & & \overline{Apr}(V) \\ & & \downarrow u" \\ \overline{Apr}(T) & \underset{\beta}{\to} & \overline{Apr}(T') \to 0 \end{array}$$

with β is surjective rough module homomorphism. We obtain:

$$\begin{array}{ccc} \overline{Apr}(L) & \xrightarrow{p} & \overline{Apr}(V) \\ q \downarrow & & \downarrow u" \end{array}$$

$$\overline{Apr}(T) \overset{\beta}{\rightarrow} \overline{Apr}(T').$$

Since β is surjective and let the kernel of p is $\overline{Apr}(L')$, there exists an exact sequence

$$0 \rightarrow \overline{Apr}(L') \rightarrow \overline{Apr}(L) \overset{p}{\rightarrow} \overline{Apr}(V) \rightarrow 0.$$

We know that the exact sequence is split, then there exists $v: \overline{Apr}(V) \rightarrow \overline{Apr}(L)$ such that $p \circ v = id_p$. So, $u = q \circ v$ is an homomorphism from $\overline{Apr}(V)$ to $\overline{Apr}(T)$ and $\beta \circ u = \beta \circ q \circ v = u'' \circ p \circ v = u''$. We can conclude that the rough module $\overline{Apr}(V)$ is a rough projective module.

Lemma 1. *Rough ring $\overline{Apr}(R)$ is the rough projective module over rough ring $\overline{Apr}(R)$ [6].*

Example 3. From **Example 1**, we have $\overline{Apr}(S), \overline{Apr}(T), \overline{Apr}(V)$ is rough module. There exists surjective rough module homomorphism defined by $f: \overline{Apr}(S) \rightarrow \overline{Apr}(T)$ with f is an identity function, for every $a \in \overline{Apr}(S)$, $h: \overline{Apr}(V) \rightarrow \overline{Apr}(T)$ with $h(a) = a \bmod 80$, for every $a \in \overline{Apr}(V)$, and $g: \overline{Apr}(S) \rightarrow \overline{Apr}(V)$ with $g(a) = a \bmod 80$ for every $\overline{Apr}(S)$. Therefore, g is well defined and $f = h \circ g$.

3.2 The Properties of the Rough Projective Module

The external direct sum is the direct multiplication of the module. The external direct sum also applies to the rough projective module. We give the property of the external direct sum of the rough projective module in the following proposition.

Proposition 1. *Let Given $M_1, M_2, \dots, M_n$ be the rough projective module in an approximation space (U, θ). $M_1 \oplus M_2 \oplus \dots \oplus M_n$ is the rough projective module in the approximation space (U^n, θ^n) with $(a_1, a_2, \dots, a_n)\theta^n(b_1, b_2, \dots, b_n)$ if only if $(a_1\theta b_1, a_2\theta b_2, \dots, a_n\theta b_n)$ for every $(a_1, \dots, a_n), (b_1, \dots, b_n) \in M_1 \oplus M_2 \oplus \dots \oplus M_n$.*

Proof. First, we show that θ^n is an equivalence relation in U^n.

1. Given $(a_1, a_2, \dots, a_n) \in U^n$, then $(a_1, a_2, \dots, a_n)\theta^n(a_1, a_2, \dots, a_n)$. So, θ^n is reflexive.
2. Given $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in U^n$ with $(a_1, a_2, \dots, a_n)\theta^n(b_1, b_2, \dots, b_n)$, then $(b_1, b_2, \dots, b_n)\theta^n(a_1, a_2, \dots, a_n)$. So, θ^n is symmetric.
3. Given $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n), (c_1, c_2, \dots, c_n) \in U^n$ with $(a_1, a_2, \dots, a_n)\theta^n(b_1, b_2, \dots, b_n)$ and $(b_1, b_2, \dots, b_n)\theta^n(c_1, c_2, \dots, c_n)$, then $(a_1, a_2, \dots, a_n)\theta^n(c_1, c_2, \dots, c_n)$. So, θ^n is transitive.

So, θ^n is an equivalence relation in U^n.

Next, we will show $M_1 \oplus M_2 \oplus \dots \oplus M_n$ is a rough module in approximation space (U^n, θ^n).
We know $\langle M_1, + \rangle, \langle M_2, + \rangle, \dots, \langle M_n, + \rangle$ are rough modules defined by binary operation $M_1 \oplus M_2 \oplus \dots \oplus M_n$ is $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 +_1 b_1, a_2 +_2 b_2, \dots, a_n +_n b_n)$.

1. Given any $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \dots \oplus \overline{Apr}(M_n)$ and $r \in \overline{Apr}(R)$ then $r\big((a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)\big) = r(a_1, a_2, \dots, a_n) + r(b_1, b_2, \dots, b_n)$.
2. Given any $(a_1, a_2, \dots, a_n) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \dots \oplus \overline{Apr}(M_n)$ and $r_1, r_2 \in \overline{Apr}(R)$, then $(r_1 + r_2)(a_1, a_2, \dots, a_n) = r_1(a_1, a_2, \dots, a_n) + r_2(a_1, a_2, \dots, a_n)$.
3. Given any $(a_1, a_2, \dots, a_n) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \dots \oplus \overline{Apr}(M_n)$ and $r_1, r_2 \in \overline{Apr}(R)$, then $(r_1 \cdot r_2)(a_1, a_2, \dots, a_n) = r_1\big(r_2(a_1, a_2, \dots, a_n)\big)$ and $(r_1 \cdot r_2)(a_1, a_2, \dots, a_n) = r_2\big(r_1(a_1, a_2, \dots, a_n)\big)$.
4. Given any $(a_1, a_2, \dots, a_n) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \dots \oplus \overline{Apr}(M_n)$ and $1 \in \overline{Apr}(R)$, then $1(a_1, a_2, \dots, a_n) = a_1, a_2, \dots, a_n$.

Hence, $M_1 \oplus M_2 \oplus \dots \oplus M_n$ is rough module in approximation space (U^n, θ^n).

Now, we will show $M_1 \oplus M_2 \oplus \dots \oplus M_n$ is a rough projective module in approximation space U^n.
We know $M_1, M_2, \dots, M_n$ are rough projective modules.

1. Given homomorphism $\alpha: \overline{Apr}(M_1 \oplus M_2 \oplus \dots \oplus M_n) \rightarrow \overline{Apr}(A)$ and $\mu: \overline{Apr}(B) \rightarrow \overline{Apr}(A)$ with diagrams

$$\begin{array}{c} \overline{Apr}(M_1 \oplus M_2 \oplus \dots \oplus M_n) \\ \downarrow \alpha \\ \overline{Apr}(B) \underset{\mu}{\rightarrow} \overline{Apr}(A) \rightarrow 0. \end{array}$$

2. Since M_i is projective, there is homomorphism $\beta_i: \overline{Apr}(M_i) \rightarrow \overline{Apr}(B)$ and mapping $p_i: \overline{Apr}(M_i) \rightarrow \overline{Apr}(M_1 \oplus M_2 \oplus \dots \oplus M_n)$ such that $\mu \circ \beta_i = \alpha \circ p_i$. We can see this in the following diagram.

$$\begin{array}{ccc} & & \overline{Apr}(M_i) \\ & \beta_i \swarrow & \downarrow \alpha \circ p_i \\ \overline{Apr}(B) & \underset{\mu}{\rightarrow} & \overline{Apr}(A). \end{array}$$

3. There is $\beta: \overline{Apr}(M_1 \oplus M_2 \oplus \dots \oplus M_n) \rightarrow \overline{Apr}(B)$ such that $\beta_i = \beta \circ p_i$. Therefore $\mu \circ \beta_i = \alpha \circ p_i$ implies $\mu \circ \beta \circ p_i = \alpha \circ p_i$. We can see this in the following diagram.

$$\begin{array}{ccc} & & \overline{Apr}(M_1 \oplus M_2 \oplus \dots \oplus M_n) \\ & \beta \swarrow & \uparrow p_i \\ \overline{Apr}(B) & \underset{\beta_i}{\leftarrow} & \overline{Apr}(M_i). \end{array}$$

So, we can conclude that $M_1 \oplus M_2 \oplus \dots \oplus M_n$ is a rough projective module in (U^n, θ^n). □

Example 4. Let $U = \mathbb{Z}_{12} = \{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6},\overline{7},\overline{8},\overline{9},\overline{10},\overline{11}\}$, we define $x\theta y$ if only if $x - y = 2a$ with $a \in \mathbb{Z}$, for every $x, y \in U$. We have two equivalence classes in **Table 5** based on the equivalence relation.

Table 5. The equivalence classes of $\mathbb{Z}_{12}$

The Equivalence Classes	The Elements of the Equivalence Classes
E_1	$\{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8},\overline{10}\}$
E_2	$\{\overline{1},\overline{3},\overline{5},\overline{7},\overline{9},\overline{11}\}$

Given $R = \{\overline{1},\overline{2},\overline{10},\overline{11}\} \subseteq U$. Then $\overline{Apr}(R) = \mathbb{Z}_{12}$ dan $\underline{Apr}(R) = \emptyset$. Next, we define the binary operation $\langle R, +_{12}, \cdot_{12} \rangle$. We will show that R is a rough ring.

1. For every $x, y \in R$ then $x +_{12} y \in \overline{Apr}(R)$.
2. Commutative property holds binary operation in R.
3. Associative property holds binary operation $+$ in $\overline{Apr}(R)$.
4. There exist $0 \in \overline{Apr}(R)$ such that for every $x \in R$, $x +_{12} 0 = 0 +_{12} x = x$.
5. Every element of R has a rough inverse in R in **Table 6**.

Table 6. Inverse Element of *R*

$x \in R$	Inverse of x
$\overline{1}$	$\overline{11}$
$\overline{2}$	$\overline{10}$
$\overline{10}$	$\overline{2}$
$\overline{11}$	$\overline{1}$

6. For every $x, y \in R$ then $x \cdot_{12} y \in \overline{Apr}(R)$. We can see this in the **Table 7**.

Table 7. Table Cayley $\cdot_{80}$ on *R*

$\cdot_{12}$	$\overline{1}$	$\overline{2}$	$\overline{10}$	$\overline{11}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{10}$	$\overline{11}$
$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{8}$	$\overline{9}$
$\overline{10}$	$\overline{10}$	$\overline{8}$	$\overline{4}$	$\overline{2}$
$\overline{11}$	$\overline{11}$	$\overline{10}$	$\overline{2}$	$\overline{1}$

7. Associative property holds binary operation $\times$ in $\overline{Apr}(R)$.
8. Left distributive law and right distributive law property holds binary operation $\times$ in $\overline{Apr}(R)$.

So, it proves R is a rough ring in approximation space (U, θ).

Furthermore, we give five rough modules to form a rough exact sequence. Let $M_1 = \{\overline{0},\overline{1},\overline{3},\overline{6},\overline{9},\overline{11}\}$, then $\overline{Apr}(M_1) = \mathbb{Z}_{12}$.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(M_1)$ then $a \cdot_{12} (x +_{12} y) = (a \cdot_{12} x) +_{12} (a \cdot_{12} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_1)$ then $(a +_{12} b) \cdot_{12} x = (a \cdot_{12} x) +_{12} (b \cdot_{12} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_1)$ then $(a \cdot_{12} b) \cdot_{12} x = a \cdot_{12} (b \cdot_{12} x)$ and $(a \cdot_{12} b) \cdot_{12} x = b \cdot_{12} (a \cdot_{12} x)$.
4. Given $1 \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_1)$ then $\overline{1} \cdot_{12} x = x$.

So, M_1 is a rough module.

Let $M_2 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$, then $\overline{Apr}(M_2) = \mathbb{Z}_{12}$.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(M_2)$ then $a \cdot_{12} (x +_{12} y) = (a \cdot_{12} x) +_{12} (a \cdot_{12} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_2)$ then $(a +_{12} b) \cdot_{12} x = (a \cdot_{12} x) +_{12} (b \cdot_{12} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_2)$ then $(a \cdot_{12} b) \cdot_{12} x = a \cdot_{12} (b \cdot_{12} x)$ and $(a \cdot_{12} b) \cdot_{12} x = b \cdot_{12} (a \cdot_{12} x)$.
4. Given $1 \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_2)$ then $\overline{1} \cdot_{12} x = x$.

Hence, M_2 is a rough module.

Let $M_3 = \{\overline{0}, \overline{1}, \overline{2}, \overline{6}, \overline{10}, \overline{11}\}$, then $\overline{Apr}(M_3) = \mathbb{Z}_{12}$.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(M_3)$ then $a \cdot_{12} (x +_{12} y) = (a \cdot_{12} x) +_{12} (a \cdot_{12} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_3)$ then $(a +_{12} b) \cdot_{12} x = (a \cdot_{12} x) +_{12} (b \cdot_{12} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_3)$ then $(a \cdot_{12} b) \cdot_{12} x = a \cdot_{12} (b \cdot_{12} x)$ and $(a \cdot_{12} b) \cdot_{12} x = b \cdot_{12} (a \cdot_{12} x)$.
4. Given $1 \in \overline{Apr}(R)$ and $x \in \overline{Apr}(M_3)$ then $\overline{1} \cdot_{12} x = x$.

Therefore, M_3 is a rough module.

Let $A = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{6}, \overline{9}, \overline{10}, \overline{11}\}$, then $\overline{Apr}(A) = \mathbb{Z}_{12}$.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(A)$ then $a \cdot_{12} (x +_{12} y) = (a \cdot_{12} x) +_{12} (a \cdot_{12} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(A)$ then $(a +_{12} b) \cdot_{12} x = (a \cdot_{12} x) +_{12} (b \cdot_{12} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(A)$ then $(a \cdot_{12} b) \cdot_{12} x = a \cdot_{12} (b \cdot_{12} x)$ and $(a \cdot_{12} b) \cdot_{12} x = b \cdot_{12} (a \cdot_{12} x)$.
4. Given $1 \in \overline{Apr}(R)$ and $x \in \overline{Apr}(A)$ then $\overline{1} \cdot_{12} x = x$.

Hence, A is a rough module.

Let $B = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}, \overline{10}\}$, then $\overline{Apr}(B) = \mathbb{Z}_{12}$.

1. Given any $a \in \overline{Apr}(R)$ and $x, y \in \overline{Apr}(B)$ then $a \cdot_{12} (x +_{12} y) = (a \cdot_{12} x) +_{12} (a \cdot_{12} y)$.
2. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(B)$ then $(a +_{12} b) \cdot_{12} x = (a \cdot_{12} x) +_{12} (b \cdot_{12} x)$.
3. Given any $a, b \in \overline{Apr}(R)$ and $x \in \overline{Apr}(B)$ then $(a \cdot_{12} b) \cdot_{12} x = a \cdot_{12} (b \cdot_{12} x)$ and $(a \cdot_{12} b) \cdot_{12} x = b \cdot_{12} (a \cdot_{12} x)$.
4. Given $1 \in \overline{Apr}(R)$ and $x \in \overline{Apr}(B)$ then $\overline{1} \cdot_{12} x = x$.

Therefore, B is a rough module.

The homomorphisms $f_1, f_2, f_3, g_1, g_2, g_3$ are surjective rough module homomorphisms define by $f_1 : \overline{Apr}(B) \to \overline{Apr}(M_1)$ with $f_1(a) = a \bmod 12$ for every $a \in \overline{Apr}(B)$, $g_1 : \overline{Apr}(M_1) \to \overline{Apr}(B)$ with $g_1(a) =$identity function for every $a \in \overline{Apr}(M_1)$, $f_2 : \overline{Apr}(B) \to \overline{Apr}(M_2)$ with $f_2(a) = a \bmod 12$ for every $a \in \overline{Apr}(B)$, $g_2 : \overline{Apr}(M_2) \to \overline{Apr}(B)$ with $g_2(a) =$ identity function for every $a \in \overline{Apr}(M_2)$, $f_3 : \overline{Apr}(B) \to \overline{Apr}(M_3)$ with $f_3(a) = a \bmod 12$ for every $a \in \overline{Apr}(B)$, $g_3 : \overline{Apr}(M_3) \to \overline{Apr}(B)$ with $g_3(a) =$ identity function for every $a \in \overline{Apr}(M_3)$. Therefore, $f_1 \circ g_1 = id, f_2 \circ g_2 = id$, and $f_3 \circ g_3 = id$. So, it proves that $\overline{Apr}(M_1), \overline{Apr}(M_2), \overline{Apr}(M_3)$ are rough projective modules.

Next, we will show that θ^3 is the equivalence relation on the set $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

1. Given any $(x_1, x_2, x_3) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$ then $(x_1, x_2, x_3)\theta^3(x_1, x_2, x_3)$. So, θ^3 is reflexive.
2. Given any $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$ with $(x_1, x_2, x_3)\theta^3(y_1, y_2, y_3)$, then $(y_1, y_2, y_3)\theta^3(x_1, x_2, x_3)$. So, θ^3 is symmetris.
3. Given any $(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$ with $(x_1, x_2, x_3)\theta^3(y_1, y_2, y_3)$ and $(y_1, y_2, y_3)\theta^3(z_1, z_2, z_3)$ then $(x_1, x_2, x_3)\theta^3(z_1, z_2, z_3)$. So, θ^3 is transitive.

Therefore, it proves that θ^3 is the equivalence relation on the set $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

Next, we know that $M_1 = \{\overline{0},\overline{1},\overline{3},\overline{6},\overline{9},\overline{11}\}$, $M_2 = \{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8},\overline{10}\}$ and $M_3 = \{\overline{0},\overline{1},\overline{2},\overline{6},\overline{10},\overline{11}\}$. Therefore:

$$\overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \overline{Apr}(M_3) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 = \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$$

and $\underline{Apr}(M_1) \oplus \underline{Apr}(M_2) \oplus \underline{Apr}(M_3) = \emptyset$. We will show that $M_1 \oplus M_2 \oplus M_3$ is a rough module in approximation space $(\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}, \theta^3)$.

1. Given any $(a_1,a_2,a_3),(b_1,b_2,b_3) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \overline{Apr}(M_3)$ and $r \in \overline{Apr}(R)$, then $r\big((a_1,a_2,a_3),(b_1,b_2,b_3)\big) = r(a_1,a_2,a_3) + r(b_1,b_2,b_3)$.
2. Given any $(a_1,a_2,a_3) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \overline{Apr}(M_3)$ and $r_1, r_2 \in \overline{Apr}(R)$, then $(r_1 + r_2)(a_1,a_2,a_3) = r_1(a_1,a_2,a_3) + r_2(a_1,a_2,a_3)$.
3. Given any $(a_1,a_2,a_3) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \overline{Apr}(M_3)$ and $r_1, r_2 \in \overline{Apr}(R)$. Therefore, we have $(r_1r_2)(a_1,a_2,a_3) = r_1\big(r_2(a_1,a_2,a_3)\big)$ and $(r_1r_2)(a_1,a_2,a_3) = r_2\big(r_1(a_1,a_2,a_3)\big)$.
4. Given any $(a_1,a_2,a_3) \in \overline{Apr}(M_1) \oplus \overline{Apr}(M_2) \oplus \overline{Apr}(M_3)$ and $1 \in \overline{Apr}(R)$ then $1(a_1,a_2,a_3) = (a_1,a_2,a_3)$.

Hence, $M_1 \oplus M_2 \oplus M_3$ is a rough module in approximation space $(\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}, \theta^3)$.

We know that M_1, M_2, M_3 are rough projective modules. Then there is a rough homomorphism module $g_1 : \overline{Apr}(M_1) \to \overline{Apr}(B)$, $g_2 : \overline{Apr}(M_2) \to \overline{Apr}(B)$, and $g_3 : \overline{Apr}(M_3) \to \overline{Apr}(B)$. Next, we will show that $M_1 \oplus M_2 \oplus M_3$ is a rough projective module.

1. Given any homomorphism $\alpha : \overline{Apr}(M_1 \oplus M_2 \oplus M_3) \to \overline{Apr}(B)$ and $\mu : \overline{Apr}(A) \to \overline{Apr}(B)$.

$$\begin{array}{c} \overline{Apr}(M_1 \oplus M_2 \oplus M_3) \\ \downarrow \alpha \\ \overline{Apr}(A) \underset{\mu}{\to} \overline{Apr}(B) \to 0. \end{array}$$

2. M_i is projective, there is homomorphism $g_i : \overline{Apr}(M_i) \to \overline{Apr}(A)$ and $p_i : \overline{Apr}(M_i) \to \overline{Apr}(M_1 \oplus M_2 \oplus M_3)$ such that $\mu \circ g_i = \alpha \circ p_i$. We can see in the following diagram.
3.

$$\begin{array}{ccc} & & \overline{Apr}(M_i) \\ & g_i \swarrow & \downarrow \alpha \circ p_i \\ \overline{Apr}(A) & \underset{\mu}{\to} & \overline{Apr}(B). \end{array}$$

4. There is $g : \overline{Apr}(M_1 \oplus M_2 \oplus M_3) \to \overline{Apr}(A)$ such that $g_i = g \circ p_i$. Therefore, $\mu \circ g_i = \alpha \circ p_i$ implies $\mu \circ g \circ p_i = \alpha \circ p_i$ with diagrams

$$\begin{array}{ccc} & & \overline{Apr}(M_1 \oplus M_2 \oplus M_3) \\ & g \swarrow & \uparrow p_i \\ \overline{Apr}(A) & \underset{g_i}{\leftarrow} & \overline{Apr}(M_i). \end{array}$$

We can conclude that $M_1 \oplus M_2 \oplus M_3$ is arough projective module in approximation space $(\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}, \theta^3)$.

Proposition 2 *Every projective module is a rough projective module over rough ring in approximation space* (U, θ).

Proof. Given a non-empty set $P \subseteq R$, P is projective module over ring R if every exact sequence

$$0 \to M_1 \to M \to P \to 0$$

is split. Rough module $\overline{Apr}(P)$ over rough ring $\overline{Apr}(R)$ is rough projective module if every rough exact sequence $0 \to \overline{Apr}(M_1) \to \overline{Apr}(M) \to \overline{Apr}(P) \to 0$ is split. Because $P \subseteq \overline{Apr}(P)$, then it proves every projective module is rough projective module over rough ring. □

Example 5. Based on **Example 1**, $\overline{Apr}(V)$ is a rough projective module over rough ring $\overline{Apr}(R)$. Next, we will investigate whether the rough projective module $\overline{Apr}(V)$ over the rough ring $\overline{Apr}(R)$ is also a projective module over the ring and also investigate analogously whether the projective module over the ring is a rough projective module over rough ring.

Given a ring $\langle R,+,\cdot\rangle$ and $S,T,V\subseteq R$ with $S,T,V\neq\emptyset$. We have:
$R=\{\overline{0},\overline{5},\overline{15},\overline{20},\overline{30},\overline{35},\overline{45},\overline{50},\overline{55},\overline{60},\overline{65},\overline{70},\overline{75}\}$, $S=\{\overline{0},\overline{3},\overline{6},\overline{9},\overline{12},\overline{15},\overline{65},\overline{71},\overline{74},\overline{77}\}$,
$T=\{\overline{0},\overline{9},\overline{18},\overline{27},\overline{53},\overline{62},\overline{71}\}$, and $V=\{\overline{0},\overline{15},\overline{20},\overline{30},\overline{50},\overline{60},\overline{65}\}$.

Now, we will show that V is projective module over ring R. On the exact sequence $0\rightarrow S\rightarrow T\rightarrow V\rightarrow 0$, we will show there is surjective homomorphism module defined by $\psi: T\rightarrow V$ with $\psi(a)=a$, for every $a\in T$.

1. Given any $a,b\in T$, then $\psi(a+_{80}b)=\psi(a)+_{80}\psi(b)$. If we choose $9,18\in T$, then $\psi(9+_{80}18)\neq\psi(9)+_{80}\psi(18)$. Therefore, $\psi(a+_{80}b)\neq\psi(a)+_{80}\psi(b)$, for every $a,b\in T$.
2. Given any $n\in R$ and $a\in T$, then $\psi(n\cdot_{80}a)=n\psi(a)$. If we choose $5\in R, 9\in T, \psi(5\cdot_{80}9)\neq 5\psi(9)$. Therefore, $\psi(n\cdot_{80}a)\neq n\psi(a)$, for every $n\in R$ and $a\in T$.
3. Given any $5a\in V$. There is $a\in T$ such that $\psi(a)=5a$. For $9\in T$ such that $\psi(9)=45\notin V$. Therefore, ψ is not surjective.

So, ψ is not surjective module homomorphism and hence V is not projective module over ring R. Therefore, every rough projective module over rough ring not necessarily projective module.

4. CONCLUSIONS

Every rough projective module is not absolute a projective module, but every projective module is rough projective module over rough ring in approximation space (U,θ). If $M_1,M_2,\ldots,M_n$ are rough projective module in approximation space (U,θ), then $M_1\oplus M_2\oplus\ldots\oplus M_n$ is a rough projective module in approximation space (U^n,θ^n) within $(a_1,a_2,\ldots,a_n)\theta^n(b_1,b_2,\ldots,b_n)$ if and only if $(a_1\theta b_1,a_2\theta b_2,\ldots,a_n\theta b_n)$ for every $(a_1,a_2,\ldots,a_n),(b_1,b_2,\ldots,b_n)\in M_1\oplus M_2\oplus\ldots\oplus M_n$.

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