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TOTAL EDGE IRREGULAR LABELING FOR TRIANGULAR GRID GRAPHS AND RELATED GRAPHS

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ABSTRACT

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph with V_Γ and E_Γ are the set of its vertices and edges, respectively. Total edge irregular k-labeling on Γ is a map from $V_\Gamma \cup E_\Gamma$ to $\{1,2,\dots,k\}$ satisfies for any two distinct edges have distinct weights. The minimum k for which the Γ satisfies the labeling is spoken as its strength of total edge irregular labeling, represented by $tes(\Gamma)$. In this paper, we discuss the tes of triangular grid graphs, its spanning subgraphs, and Sierpiński gasket graphs.

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1. INTRODUCTION

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a simple, undirected, and connected graph where V_Γ and E_Γ are the set of vertices and edges of Γ, respectively. A map that assigns some set of elements of graph Γ to the set of positive or non-negative integers is spoken as labeling. The domain of this map can only be the set of vertices (*vertex labeling*), the set of edges (*edge labeling*), or the union of vertex and edge set (*total labeling*) [1].

In this paper, we only discuss about particular case of total labeling i.e. *total edge irregular k-labeling*. In mathematical word, graph Γ is a total edge irregular k-labeling graph if there exists a map $\phi : V_\Gamma \cup E_\Gamma \to \{1,2,\dots,k\}$ such that for any $ab, cd \in E_\Gamma$, $wt_\phi(ab) \neq wt_\phi(cd)$. We called $wt_\phi(ab)$ as the weight of edge ab and it is defined as $wt_\phi(ab) = \phi(a) + \phi(ab) + \phi(b)$. The minimum k for which ϕ exists is spoken as the strength of total edge irregular labeling of Γ, represented by $tes(\Gamma)$. Let $\Delta(\Gamma)$ be a maximum vertex degree of Γ. Bača, et al. [1] showed that the *tes* of any given graph Γ is at least

$$\max\left\{\left\lceil\frac{|E_\Gamma|+2}{3}\right\rceil, \left\lceil\frac{\Delta(\Gamma)+1}{2}\right\rceil\right\}$$

In fact, all graphs are conjectured by Ivančo and Jendrol in [2] to have total edge irregularity that is equal to the lower bound i.e.

$$tes(\Gamma) = \max\left\{\left\lceil\frac{|E_\Gamma|+2}{3}\right\rceil, \left\lceil\frac{\Delta(\Gamma)+1}{2}\right\rceil\right\} \tag{1}$$

Some authors have showed that **Equation (1)** is true for certain graphs, such as trees [2], path and cycle graphs [1], some cycle related graphs [3], large graphs [4], certain family of graphs [5], complete graphs and complete bipartite graphs [6], zigzag graphs [7], disjoint union of wheel graphs [8], centralized uniform theta graphs [9], book and double book graphs [10], triple book graphs [11], polar grid graph [12], staircase graphs and related graphs [13], and generalized arithmetic staircase graphs [14], and ladder-related graphs [15].

Let $x, y \in \mathbb{R}$, so that $x \leq \lceil x \rceil$ and $y \leq \lceil y \rceil$. Then

$$x + y \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil \tag{2}$$

and

$$n + \lceil x \rceil = \lceil n + x \rceil \tag{3}$$

$$n - \lceil x \rceil = \lceil n - x \rceil - 1 \tag{4}$$

for all $n \in \mathbb{Z}$.

In this paper, we will show that **Equation (1)** is also true for three types of graphs, namely for triangular grid graphs, some spanning subgraphs of triangular grid graph, and some Sierpiński gasket graphs.

2. RESULTS AND DISCUSSION

In this section, we will discuss triangular grid graph and its spanning subgraphs and Sierpiński gasket graphs, from the terminology of each graph up to the result on their total edge irregularity strength. In particular, by proving *tes* of triangular grid graph and its spanning subgraph, we do explain in a certain way to get the labels by seeing the structure of those graphs.

2.1 Triangular Grid Graphs

Triangular grid graph $T_n = \left(V_{T_n}, E_{T_n}\right)$ of n levels is a graph obtained by piling up $\frac{n(n+1)}{2}$ cycles of length 3 such that it forms a bigger triangle (see **Figure 1**). Formally, T_n has

$$V_{T_n} = \left\{v_{0,0}, v_{i,1}, \dots, v_{i,i+1}\ ;\ i = 1,2,\dots,n\right\}$$

and E_{T_n} which contains of edges as follows.

$$E_{T_n} = \{v_{0,0}v_{1,1}, v_{0,0}v_{1,2}\} \cup \{v_{i,j}v_{i,j+1}\ ;\ i = 1,2,\dots,n; j = 1,2,\dots,i\}$$
$$\cup \{v_{i,j}v_{i+1,j}\ ;\ i = 1,2,\dots,n-1; j = 1,2,\dots,i+1\} \cup \{v_{i,j}v_{i+1,j+1}\ ;\ i = 1,2,\dots,n-1; j = 1,2,\dots,i+1\}$$

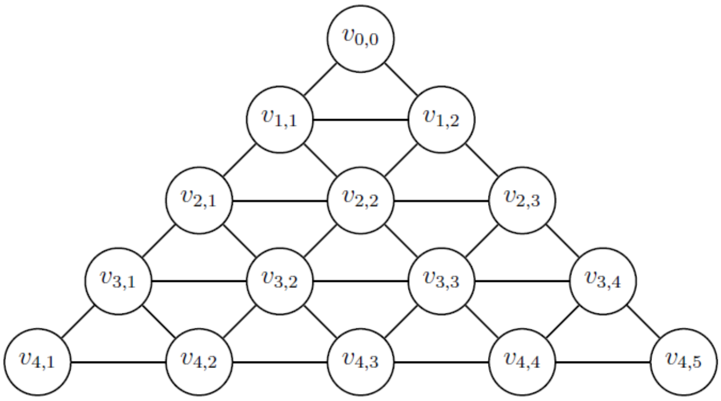

Figure 1. Triangular grid graph T_4

On graph T_n, we divide the edges into three types such as horizontal edge, right diagonal edge, and left diagonal edge. The horizontal edges are the elements of the set $\{v_{i,j}v_{i,j+1} : i = 1,2,\dots,n; j = 1,2,\dots,i\}$, the right diagonal edges are the elements of the set $\{v_{i,j}v_{i+1,j}\ ; i = 1,2,\dots,n-1; j = 1,2,\dots,i+1\}$, and left diagonal edges are the elements of the set $\{v_{i,j}v_{i+1,j+1}\ ;\ i = 1,2,\dots,n-1; j = 1,2,\dots,i+1\}$. By **Figure 1**, clearly there are n horizontal edges and $2n$ diagonal (right and left) edges of each level. We calculate that for arbitrary $n \geq 1$, we have the number of vertices and edges of T_n are $\frac{(n+1)(n+2)}{2}$ and $\frac{3n(n+1)}{2}$, respectively. For $i = 2,3,\dots,n$, let d_i be the weight of the last diagonal edges at the ith level and let h_i be the weight of the last horizontal edges at the ith level. In graph T_n, we have $d_i = wt_\phi(v_{i-1,i}v_{i,i+1})$ and $h_i = wt_\phi(v_{i,i}v_{i,i+1})$. These terms may be important to prove the tes of any given graph especially triangular grid graphs and related graphs. To do that, we first determine an explicit formula of d_i and h_i by using h_k (prescribed) where $k < i$, and then use d_i and h_i to find the weights and labels of every edge at ith level. The index i for d_i and h_i are depending on the regular pattern labels appear at the first time such that it might be distinct for every graph. The following theorem describes the total edge irregularity strength of T_n for any $n \in \mathbb{N}$.

Theorem 1. *For every positive integer n, it follows that $tes(T_n) = \left\lceil \frac{3n(n+1)+4}{6} \right\rceil$.*

Proof. It is easy to check that for all $n \in \mathbb{N}$, $\Delta(T_n) \leq \left\lceil \frac{3n(n+1)+4}{6} \right\rceil$. Therefore, we obtain that $tes(T_n) \geq \left\lceil \frac{3n(n+1)+4}{6} \right\rceil$. To prove **Equation (1)**, we have to show that $tes(T_n) \leq \left\lceil \frac{3n(n+1)+4}{6} \right\rceil$ for any $n \in \mathbb{N}$, by showing that there exists a map $\phi : V(T_n) \cup E(T_n) \to \left\{1,2,\dots,\left\lceil \frac{3n(n+1)+4}{6} \right\rceil\right\}$. For $n \geq 2$, we label all vertices by $\phi(v_{i,j}) = \left\lceil \frac{3i(i+1)+4}{6} \right\rceil$, where $i = 2,3,\dots,n$. Consequently, the distinction of every two consecutive weights at the same level only depend on the distinction of their edge labels which are equal to 1.

For $n = 2$, we prescribe a total edge irregular 4-labeling for T_2 by **Figure 2**. We know that $h_2 = 11$, so that for $n = 3$ we have the weights of diagonal and horizontal edges provided in

Table 1.

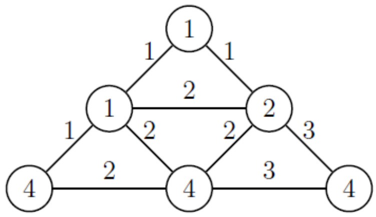

Figure 2. Graph T_2 with a total edge irregular 4-labeling

Table 1. Weight of edges

Right Diagonal Edges	Left Diagonal Edges	Horizontal Edges
$wt_\phi(v_{2,1}v_{3,1}) = h_2 + 1$	$wt_\phi(v_{2,1}v_{3,2}) = h_2 + 2$	$wt_\phi(v_{3,1}v_{3,2}) = h_2 + 7$
$wt_\phi(v_{2,2}v_{3,2}) = h_2 + 3$	$wt_\phi(v_{2,2}v_{3,3}) = h_2 + 4$	$wt_\phi(v_{3,2}v_{3,3}) = h_2 + 8$
$wt_\phi(v_{2,3}v_{3,3}) = h_2 + 5$	$wt_\phi(v_{2,3}v_{3,4}) = h_2 + 6$	$wt_\phi(v_{3,3}v_{3,4}) = h_2 + 9$

From

Table 1, we obtain $d_3 = h_2 + 6$ and $h_3 = h_2 + 9$. If we continue this observation for $n \geq 4$, we will obtain $d_i = \frac{i(3i+1)+4}{2}$ and $h_i = \frac{3i(i+1)+4}{2}$ for $i = 3,4,\dots,n$. Those formulas hold for all $i = 3,4,\dots,n$ by induction. Now, we consider the following two cases.

a. Since there are i horizontal edges at ith level, then we have $wt_\phi(v_{i,j}v_{i,j+1}) = h_i - (i-j)$, where $i = 3,4,\dots,n$ and $j = 1,2,\dots,i$. Therefore, we obtain the edge label $\phi(v_{i,j}v_{i,j+1}) = h_i - 2\phi(v_{i,j}) - (i-j)$.

b. Since there are $2i$ diagonal edges at ith level, then we have $wt_\phi(v_{i-1,j}v_{i,j}) = d_i - (2i - (2j-1))$ and $wt_\phi(v_{i-1,j}v_{i,j+1}) = d_i - (2i-2j)$, where $i = 3,4,\dots,n$ and $j = 1,2,\dots,i$. Therefore, we obtain the right diagonal edge label $\phi(v_{i-1,j}v_{i,j}) = d_i - \phi(v_{i-1,j}) - \phi(v_{i,j}) - (2i-(2j-1))$ and the left diagonal edge label $\phi(v_{i-1,j}v_{i,j+1}) = d_i - \phi(v_{i-1,j}) - \phi(v_{i,j+1}) - (2i-2j)$.

Clearly all weights are distinct and we realized that the last diagonal and horizontal edge label of ith level ($i = 3,4,\dots,n$) are always less than $\left\lceil \frac{3i(i+1)+4}{6} \right\rceil$ because of the following results.

a. Last horizontal edge label

$$\phi(v_{i,i}v_{i,i+1}) = \frac{3i(i+1)+4}{2} - 2\left\lceil \frac{3i(i+1)+4}{6} \right\rceil = \frac{i(i+1)}{2} < \frac{i(i+1)}{2} + 1 = \left\lceil \frac{3i(i+1)+4}{6} \right\rceil$$

b. Last diagonal edge label

$$\phi(v_{i-1,i}v_{i,i+1}) = \frac{i(3i+1)+4}{2} - \left\lceil \frac{3(i-1)i+4}{6} \right\rceil - \left\lceil \frac{3i(i+1)+4}{6} \right\rceil = \frac{i(i+1)}{2} < \frac{i(i+1)}{2} + 1 = \left\lceil \frac{3i(i+1)+4}{6} \right\rceil$$

Hence, the proof is completed. ■

Figure 3 illustrates a graph T_4 with a total edge irregular 11-labeling.

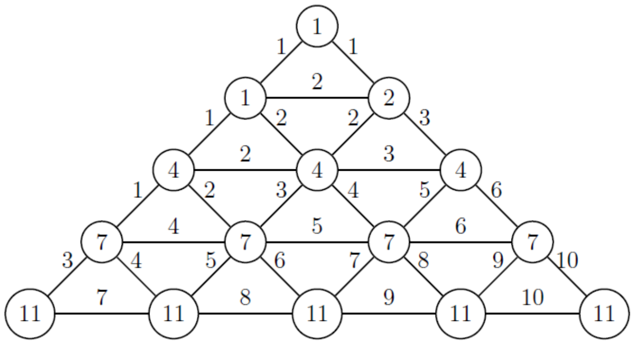

Figure 3. **Graph T_4 with a total edge irregular 11-labeling**

2.2 Spanning Subgraphs of Triangular Grid Graph

Now we come to the first spanning subgraph of triangular grid graph. This graph is a triangular grid graph without two border edges of each level, denoted by B_2T_n for all positive integer n (see **Figure 4**). Clearly $V_{B_2T_n} = V_{T_n}$ and $E_{B_2T_n} = E_{T_n} \backslash \{v_{i-1,1}v_{i,1}, v_{i-1,i}v_{i,i+1} \,;\, i = 2,3, \dots, n\}$. Therefore for $n \in \mathbb{N}$, we have $|V_{B_2T_n}| = |V_{T_n}|$ and $|E_{B_2T_n}| = |E_{T_n}| - |\{v_{i-1,1}v_{i,1}, v_{i-1,i}v_{i,i+1} \,;\, i = 2, \dots, n\}| = \frac{n(3n-1)+4}{2}$.

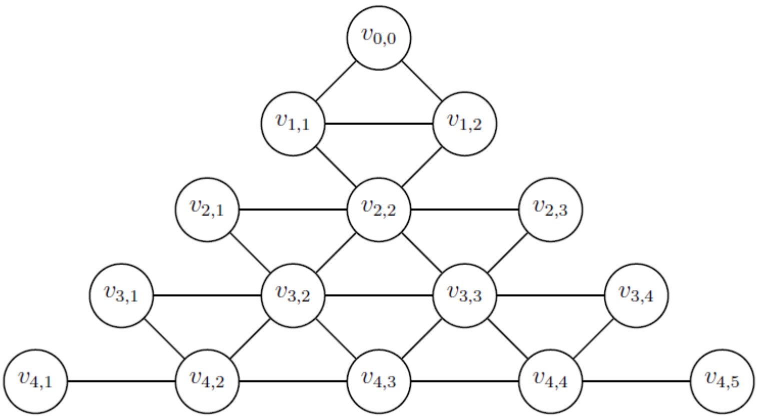

Figure 4. **Graph B_2T_4**

In this part, the **Theorem 2** and **Theorem 3** will be proved by using the terms d_i and h_i such like the previous subsection.

Theorem 2. *For every positive integer n, it follows that* $tes(B_2T_n) = \left\lceil \frac{n(3n-1)+8}{6} \right\rceil$.

Proof. It is easy to check that for any $n \in \mathbb{N}$, $\Delta(B_2T_n) \le \left\lceil \frac{n(3n-1)+8}{6} \right\rceil$. Therefore, we obtain $tes(B_2T_n) \ge \left\lceil \frac{n(3n-1)+8}{6} \right\rceil$. To prove **Equation (1)**, we have to show that $tes(T_n) \le \left\lceil \frac{n(3n-1)+8}{6} \right\rceil$ for any $n \in \mathbb{N}$, by showing that there exists a map $\phi : V_{B_2T_n} \cup E_{B_2T_n} \to \left\{1,2, \dots, \left\lceil \frac{n(3n-1)+8}{6} \right\rceil\right\}$. For $n \ge 2$, we label all vertices by $\phi(v_{i,j}) = \left\lceil \frac{i(3i-1)+8}{6} \right\rceil$, where $i = 2,3, \dots, n$. Consequently, the distinction of every two consecutive weights at the same level only depend on the distinction of their edge labels which are equal to 1. For $n = 2$, we prescribe a total edge irregular 2-labeling by **Figure 5**.

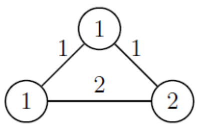

Figure 5. **Graph T_1 with total edge irregular 2-labeling**

We know that $h_2 = 9$. By doing some observations for such $n \geq 4$, we will obtain $d_i = \frac{3i(i-1)+8}{2}$ and $h_i = \frac{i(3i-1)+8}{2}$ for $i = 3,4,\dots,n$. By induction, those formulas hold for all $i = 3,4,\dots,n$. We obtain the same edge labels with the case of graph T_n but only distinct at index i and j i.e.

a. The weights of horizontal edge at ith level is $wt_\phi(v_{i,j}v_{i,j+1}) = h_i - (i-j)$ where $i = 3,4,\dots,n$ and $j = 1,2,\dots,i$. Therefore the label of horizontal edge at ith level is $\phi(v_{i,j}v_{i,j+1}) = h_i - 2\phi(v_{i,j}) - (i-j)$ where $i = 3,4,\dots,n$ and $j = 1,2,\dots,i$.

b. The weights of right diagonal edge at ith level is $wt_\phi(v_{i-1,j}v_{i,j}) = d_i - (2i-2j)$ where $i = 3,4,\dots,n$ and $j = 2,3,\dots,i$, such that the label of right diagonal edge at ith level is $\phi(v_{i-1,j}v_{i,j}) = d_i - \phi(v_{i-1,j}) - \phi(v_{i,j}) - (2i-2j)$ where $i = 3,4,\dots,n$ and $j = 2,3,\dots,i$. On the other hand, the weight of left diagonal edge at ith level is $wt_\phi(v_{i-1,j}v_{i,j+1}) = d_i - (2i-(2j-1))$ where $i = 3,4,\dots,n$ and $j = 2,3,\dots,i$, such that the label of left diagonal edge at ith level is $\phi(v_{i-1,j}v_{i,j+1}) = d_i - \phi(v_{i-1,j}) - \phi(v_{i,j+1}) - (2i-(2j-1))$ where $i = 3,4,\dots,n$ and $j = 2,3,\dots,i$.

Clearly all weights are distinct and we realized that the last diagonal and horizontal edge label of ith level ($i = 3,4,\dots,n$) are always less than $\left\lceil \frac{i(3i-1)+8}{6} \right\rceil$ because of the following results.

a. Last horizontal edge label

Suppose that $\phi(v_{i,i}v_{i,i+1}) > \left\lceil \frac{i(3i-1)+8}{6} \right\rceil$. We have

$$\frac{i(3i-1)+8}{6} > \left\lceil \frac{i(3i-1)+8}{6} \right\rceil \tag{5}$$

The **Inequality (5)** contradicts the fact that the ceiling of any real number x is greater than or equal to x. Therefore, we obtain $\phi(v_{i,i}v_{i,i+1}) \leq \left\lceil \frac{i(3i-1)+8}{6} \right\rceil$.

b. Last diagonal edge label

$$\begin{aligned}\phi(v_{i-1,i}v_{i,i}) &= \frac{3i^2-3i+8}{2} - \left\lceil \frac{3i^2-7i+12}{6} \right\rceil - \left\lceil \frac{3i^2-i+8}{6} \right\rceil \\ &\leq \frac{3i^2-3i+8}{2} - \left\lceil \frac{3i^2-7i+12}{6} + \frac{3i^2-i+8}{6} \right\rceil\end{aligned}$$

By **Equation (4)**, we have $\phi(v_{i-1,i}v_{i,i}) \leq \left\lceil \frac{3i^2-i+4}{6} \right\rceil - 1 < \left\lceil \frac{3i^2-i+8}{6} \right\rceil$.

Hence, the proof is completed. ■

Figure 6 illustrates a graph B_2T_5 with a total edge irregular 13-labeling.

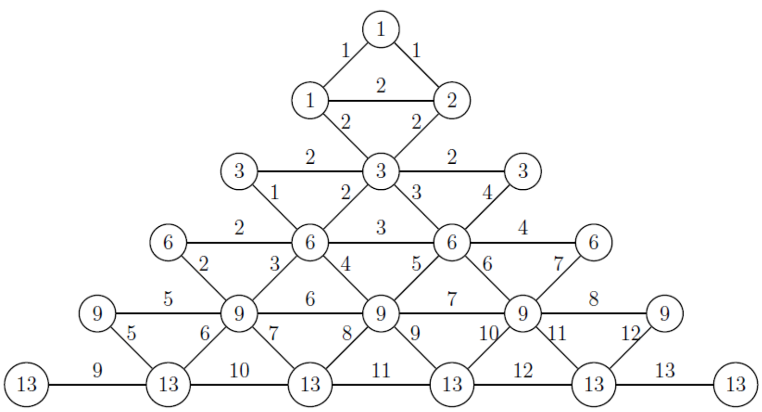

Figure 6. Graph B_2T_5 with a total edge irregular 13-labeling

The second spanning subgraph of triangular grid graph that we observe is also T_n with some modifications. We remove one border edge of each level such that there does not exist a pair of two incidence border edges which are removed together. This graph is denoted by B_1T_n. It is clear that $\left|V_{B_1T_n}\right| = \left|V_{T_n}\right|$ and $\left|E_{B_1T_n}\right| = \left|E_{T_n}\right| - n = \frac{n(3n+1)}{2}$ for $n \in \mathbb{N}$ (see **Figure 7**). Obviously, the vertex set is $V_{B_1T_n} = V_{T_n}$ and the edge set is $E_{B_1T_n} = E_{T_n} \backslash \left(\{v_{0,0}v_{1,2}, v_{i-1,i}v_{i,i+1}\ ;\ i \text{ odd}\} \cup \{v_{i-1,1}v_{i,1}\ ;\ i \text{ even}\}\right)$. Another possibility of describing B_1T_n is by defining the edge set

$$E_{B_1T_n} = E_{T_n} \backslash \left(\{v_{i-1,i}v_{i,i+1}\ ;\ i \text{ even}\} \cup \{v_{0,0}v_{1,1}, v_{i-1,1}v_{i,1}\ ;\ i \text{ odd}\}\right),$$

which we call as the mirror of graph B_1T_4 as shown at **Figure 8**.

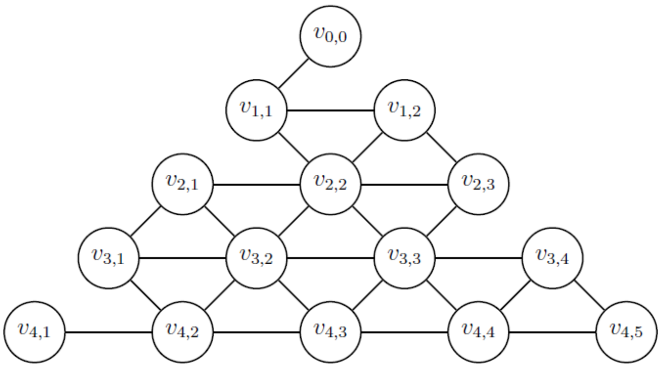

Figure 7. Graph B_1T_4

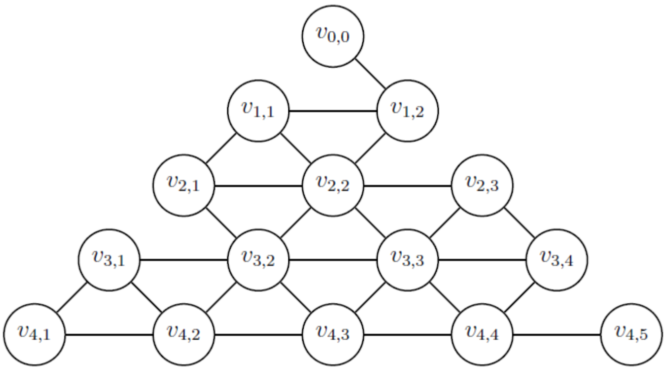

Figure 8. The mirror of graph B_1T_4

Theorem 3. *For every positive integer n, it follows that $tes(B_1T_n) = \left\lceil \frac{n(3n+1)+4}{6} \right\rceil$.*

Proof. It is easy to check that for any $n \in \mathbb{N}$, $\Delta(B_1T_n) \leq \left\lceil \frac{n(3n+1)+4}{6} \right\rceil$. Therefore, we obtain $tes(B_2T_n) \geq \left\lceil \frac{n(3n+1)+4}{6} \right\rceil$. To prove **Equation (1)**, we have to show that $tes(T_n) \leq \left\lceil \frac{n(3n+1)+4}{6} \right\rceil$ for any $n \in \mathbb{N}$, by showing that there exists a map $\phi : V(B_1T_n) \cup E(B_1T_n) \to \left\{1,2, \ldots, \left\lceil \frac{n(3n+1)+4}{6} \right\rceil\right\}$. For $n \geq 2$, we label all vertices by $\phi(v_{i,j}) = \left\lceil \frac{i(3i+1)+4}{6} \right\rceil$, where $i = 4,5, \ldots, n$. For $n = 4$, we prescribe a total edge irregular 10-labeling by **Figure 9**.

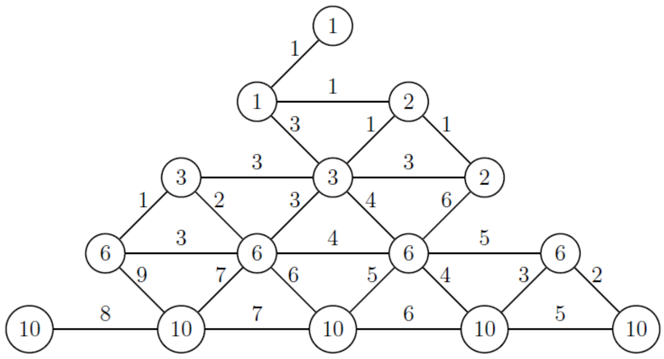

Figure 9. Graph B_1T_5 with a total edge irregular 10-labeling

We know that $h_4 = 28$. By doing some observations for $n \geq 6$, we will obtain d_i and h_i as follows

$$d_i = \begin{cases} wt_\phi(v_{i-1,1}v_{i,2}) = \dfrac{i(3i-1)+4}{2}, & i \text{ even} \\ wt_\phi(v_{i-1,i}v_{i,i}) = \dfrac{i(3i-1)+4}{2}, & i \text{ odd} \end{cases}$$

$$h_i = \begin{cases} wt_\phi(v_{i,1}v_{i,2}) = \dfrac{i(3i+1)+4}{2}, & i \text{ even} \\ wt_\phi(v_{i,i}v_{i,i+1}) = \dfrac{i(3i+1)+4}{2}, & i \text{ odd} \end{cases}$$

where $i = 5,6,\dots,n$. Those formulas hold for all $i = 5,6,\dots,n$ by induction. Now, we consider the following two cases:

a. Since there are i horizontal edges at ith level $(i = 5,6,\dots,n)$, then we have

$$wt_\phi(v_{i,j}v_{i,j+1}) = h_i - j,$$

so that the label is

$$\phi(v_{i,j}v_{i,j+1}) = h_i - 2\phi(v_{i,j}) - j,$$

where $j = 1,2,\dots,i$.

b. Let $a = \begin{cases} 1, & i \text{ even} \\ 0, & i \text{ odd} \end{cases}$ and $b = \begin{cases} 0, & i \text{ even} \\ 1, & i \text{ odd} \end{cases}$ for $i = 5,6,\dots,n$. Since there are $2i-1$ diagonal edges at ith level, then we have the weights of right diagonal edge is

$$wt_\phi(v_{i-1,j}v_{i,j}) = d_i - 2j + 3a - 2ib,$$

so that the label is

$$\phi(v_{i-1,j}v_{i,j}) = d_i - 2j + 3a - 2ib - \phi(v_{i,j}) - \phi(v_{i-1,j}),$$

where $j \in \begin{cases} \{2,3,\dots,i\}, & i \text{ even} \\ \{1,2,\dots,i\}, & i \text{ odd} \end{cases}$. On the other hand, the weights of left diagonal edge is

$$wt_\phi(v_{i-1,j}v_{i,j+1}) = d_i - 2(-1)^i j - 2a + (1-2i)b,$$

so that the label is

$$\phi(v_{i-1,j}v_{i,j+1}) = d_i - 2(-1)^i j - 2a + (1-2i)b - \phi(v_{i-1,j}) - \phi(v_{i,j+1}),$$

where $j \in \begin{cases} \{2,3,\dots,i\}, & i \text{ even} \\ \{1,2,\dots,i-1\}, & i \text{ odd} \end{cases}$. Clearly all weights are distinct and we realized that the last diagonal and horizontal edge label of ith level are always less than $\left\lceil \frac{i(3i+1)+4}{6} \right\rceil$ because of the following results.

- Last horizontal edge label
  Suppose that $\phi(v_{i,i}v_{i,i+1}) > \left\lceil \frac{i(3i+1)+4}{6} \right\rceil$ where $i = 5,6,\dots,n$. Then

$$\phi(v_{i,j}v_{i,j+1}) = \frac{i(3i+1)+4}{2} - 2\left\lceil\frac{i(3i+1)+4}{6}\right\rceil > \left\lceil\frac{i(3i+1)+4}{6}\right\rceil$$
$$\frac{i(3i+1)+4}{6} > \left\lceil\frac{i(3i+1)+4}{6}\right\rceil. \quad (6)$$

The **Inequality (6)** contradicts the fact that the ceiling of any real number x is greater than or equal to x. Therefore, we obtain $\phi(v_{i,i}v_{i,i+1}) \leq \left\lceil\frac{i(3i+1)+4}{6}\right\rceil$ where $i = 5,6,\dots,n$.

- Last diagonal edge label
  For i is even,

$$\begin{aligned}\phi(v_{i-1,1}v_{i,2}) &= d_i - \phi(v_{i-1,1}) - \phi(v_{i,2})\\ &= \frac{i(3i-1)+4}{2} - \left\lceil\frac{(i-1)(3(i-1)+1)+4}{6}\right\rceil - \left\lceil\frac{i(3i+1)+4}{6}\right\rceil\\ &= \frac{3i^2-i+4}{2} - \left(\left\lceil\frac{3i^2-5i+6}{6}\right\rceil + \left\lceil\frac{3i^2+i+4}{6}\right\rceil\right).\end{aligned}$$

By **Inequality (2)** and **Equation (4)**, then $\phi(v_{i-1,1}v_{i,2}) < \left\lceil\frac{3i^2+i-4}{6}\right\rceil < \left\lceil\frac{3i^2+i+4}{6}\right\rceil$. By the same way, we have $\phi(v_{i-1,1}v_{i,2}) < \left\lceil\frac{3i^2+i+4}{6}\right\rceil$ for i is odd.

Hence, the proof is completed. ■

The *tes* of the mirror of B_1T_n is clearly equivalent to the **Theorem 3**. The vertex labels are similar with B_1T_n but the edge labels are different with B_1T_n, precisely it is different at index i and j. Explicitly, we obtain horizontal edge label is

$$\phi(v_{i,j}v_{i,j+1}) = \phi(v_{i,j}v_{i,j+1}) = h_i - 2\phi(v_{i,j}) - j,$$

where $j = 1,2,\dots,i$. For $j \in \begin{cases}\{2,3,\dots,i\}, & i \text{ odd}\\ \{1,2,\dots,i-1\}, & i \text{ even}\end{cases}$, then the right diagonal edge label is

$$\phi(v_{i-1,j}v_{i,j}) = d_i - 2j + 3a - 2ib - \phi(v_{i,j}) - \phi(v_{i-1,j})$$

and the left diagonal edge label is

$$\phi(v_{i-1,j}v_{i,j+1}) = d_i - 2(-1)^i j - 2a + (1-2i)b - \phi(v_{i-1,j}) - \phi(v_{i,j+1}).$$

Figure 10 illustrates a graph B_1T_7 with a total edge irregular 20-labeling.

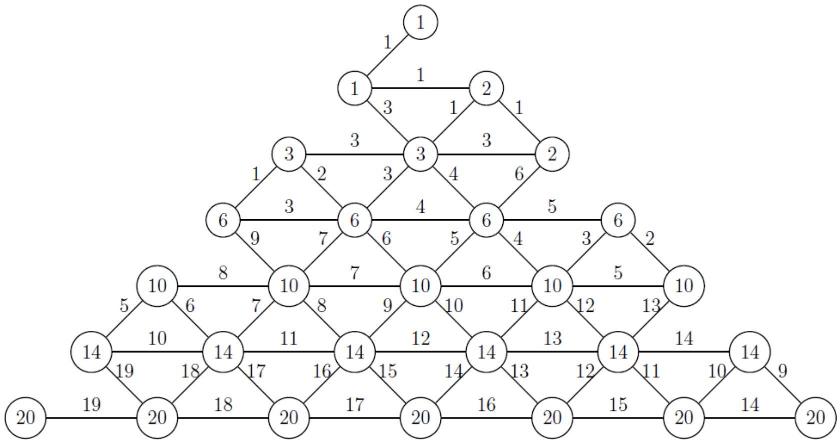

Figure 10. Graph B_1T_7 with a total edge irregular 20-labeling

2.2 Sierpiński Gasket Graphs

Sierpiński gasket is a geometric shape formed by infinitely repeated dividing a triangle into four smaller triangles out of its center, whereas Sierpiński gasket graph (SG_n) of n levels is a graph obtained by $n-1$ repeated dividing a triangle graph into smaller triangle graphs out of its center. In other words, SG_n consists of three attached copies of SG_{n-1} which refer to as top, bottom left, and bottom right components of SG_n, denoted by $SG_{n,T}$, $SG_{n,BL}$, and $SG_{n,BR}$, respectively (see **Figure 11**). It is easy to see that Sierpiński gasket graph is also a subgraph of triangular grid graph and it has $\frac{3}{2}(3^{n-1}+1)$ vertices and 3^n edges for every positive integer $n \geq 2$.

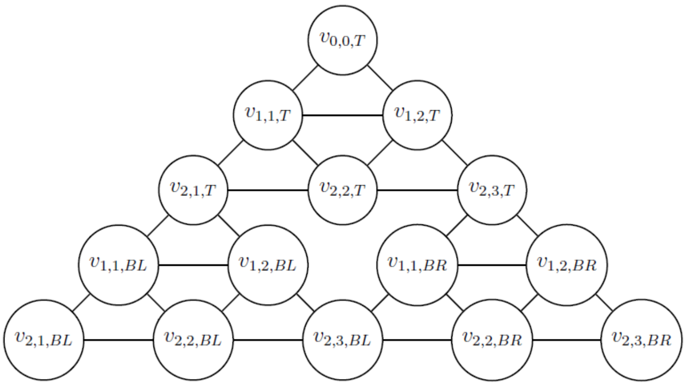

Figure 11. Sierpiński gasket graph SG_3 consists of three attached copies of SG_2 as $SG_{3,T}$, $SG_{3,BL}$, and $SG_{3,BR}$, where there are $v_{2,1,T} = v_{0,0,BL}$, $v_{2,3,T} = v_{0,0,BR}$, and $v_{2,3,BL} = v_{2,1,BR}$

The following theorem as a result of our observation about the total edge irregularity strength for some cases of Sierpiński gasket graphs.

Theorem 4. *For $n \in \{1,2,3,4\}$, it follows that $tes(SG_n) = \left\lceil \frac{3^n+2}{3} \right\rceil$.*

Proof. We will show the proof by providing a figure of labeled graph SG_n of each $n \in \{1,2,3,4\}$.

i. For $n = 1$, SG_n is isomorphic with triangular grid graph T_1 and cycle C_3. By **Figure 5** and **[1]**, we obtain $tes(SG_1) = 2$.

ii. For $n = 2$, SG_n is isomorphic with triangular grid graph T_2. By **Figure 2**, we obtain $tes(SG_2) = 4$.

iii. For $n = 3$, we obtain $tes(SG_n) = 10$ by **Figure 12**.

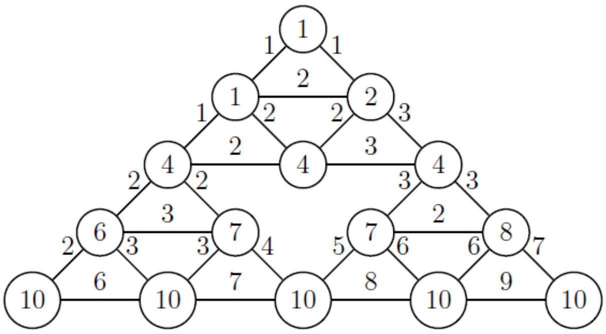

Figure 12. Graph SG_3 with total edge irregular 10-labeling

iv. For $n = 4$, we obtain $tes(SG_n) = 28$ by **Figure 13**.

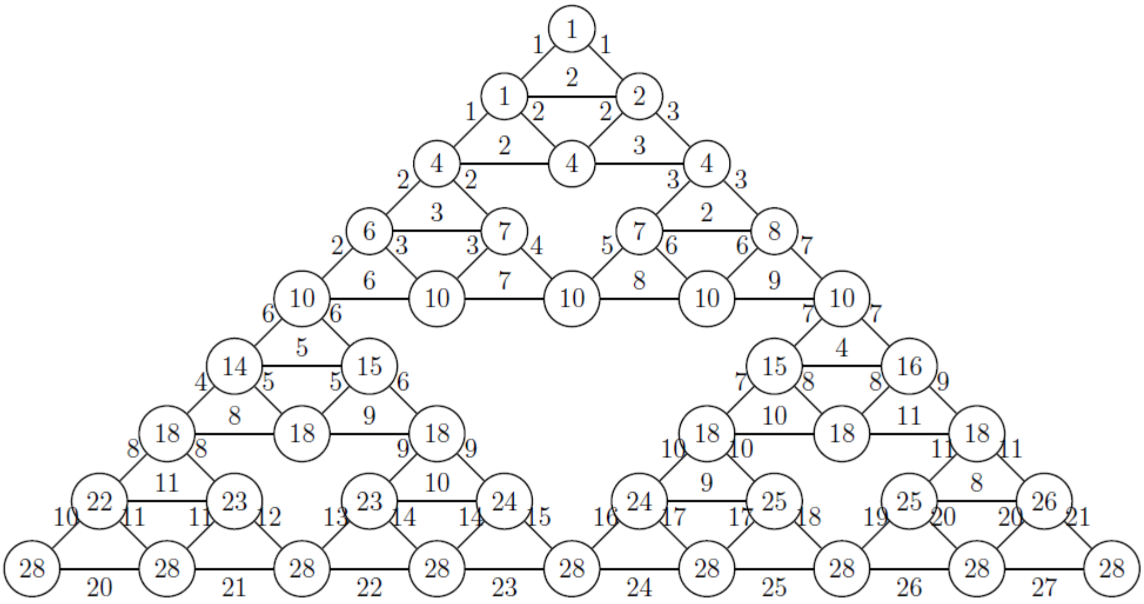

Figure 13. Graph SG_4 with total edge irregular 28-labeling

■

3. CONCLUSIONS

We conclude the *tes* of triangular grid graphs T_n for every positive integer n is $tes(T_n) = \left\lceil \frac{3n(n+1)+4}{6} \right\rceil$ and the *tes* of some spanning subgraphs of triangular grid graph i.e. B_1T_n and B_2T_n for every positive integer n are $tes(B_1T_n) = \left\lceil \frac{n(3n+1)+4}{6} \right\rceil$ and $tes(B_2T_n) = \left\lceil \frac{n(3n-1)+8}{6} \right\rceil$, respectively. In addition, the *tes* of the Sierpiński gasket graphs SG_n for $n \in \{1,2,3,4\}$ is $tes(SG_n) = \left\lceil \frac{3^n+2}{3} \right\rceil$.

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