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THE EXISTENCE OF SOLUTION OF GENERALIZED EIGENPROBLEM IN INTERVAL MAX-PLUS ALGEBRA

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ABSTRACT

An eigenproblem of a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is $A \otimes x = \lambda \otimes x$ where $\lambda \in \mathbb{R}_\varepsilon$ and $x \in \mathbb{R}_\varepsilon^n$, $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)$. Vector $x \in \mathbb{R}_\varepsilon^n$ and $\lambda \in \mathbb{R}_\varepsilon$ are eigenvector and eigenvalue, respectively. General form of eigenvalue problem is $A \otimes x = \lambda \otimes B \otimes x$ with $A, B \in \mathbb{R}_\varepsilon^{m\times n}$, $x \in \mathbb{R}_\varepsilon^n$, $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)$. Interval maks-plus algebra is $I(\mathbb{R})_\varepsilon = \{ x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \leq \overline{x} \} \cup \{\varepsilon\}$ and $\varepsilon = [\varepsilon, \varepsilon]$ equipped, with a maximum ($\overline{\oplus}$) and plus ($\overline{\otimes}$) operations. The set of $m \times n$ matrices which its component elements of $I(\mathbb{R})_\varepsilon$ is called matrices over interval max-plus algebra and denoted by $I(\mathbb{R})_\varepsilon^{m\times n}$. Let $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, eigenproblem in interval max-plus algebra is $A \overline{\otimes} x = \lambda \overline{\otimes} x$ with $x \in I(\mathbb{R})_\varepsilon^n$, $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)$ and $\lambda \in I(\mathbb{R})$. Vector $x \in I(\mathbb{R})_\varepsilon^n$ and $\lambda \in I(\mathbb{R})_\varepsilon$ are eigenvector and eigenvalue, respectively. In this research, we will discuss the generalization of the eigenproblem in interval max-plus algebra. Especially about the existence of solution of generalized eigenproblem in interval max-plus algebra.

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1. INTRODUCTION

Let $\mathbb{R}$ be a set of all real numbers and $\varepsilon = -\infty$. Max-plus algebra is a set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ that is equipped with operations maximum ($\oplus$) and plus ($\otimes$) [1], [2], [3]. Matrix over max-plus algebra is a matrix whose entries are elements of $\mathbb{R}_\varepsilon$. The set of a matrix over the max-plus algebra of size $m \times n$ denoted by $\mathbb{R}_\varepsilon^{m\times n}$ [4], [5], [6]. In the $\mathbb{R}_\varepsilon^{m\times n}$ can also be defined as operations maximum ($\oplus$) and plus ($\otimes$).

According to [7], for example, in applications on the production system, the production matrix $A = (A_{ij}) \in \mathbb{R}_\varepsilon^{n\times n}$ with A_{ij} by showing the production process time from the machine j to i, while the vector $x(k) = \left(x_i(k)\right) \in \mathbb{R}_\varepsilon^n$ with $x_i(k)$ by indicating the time of the machine i began to work at the k-th stage. In this production process, an equation is generated, $x(k+1) = A \otimes x(k)$. One of the criteria used by factory entrepreneurs (manufacturers) that the production process is expected to take place periodically with a certain period for example, so that it is obtained $x(k+1) = \lambda \otimes x(k)$. From $x(k+1) = A \otimes x(k)$ and $x(k+1) = \lambda \otimes x(k)$ obtained $A \otimes x(k) = \lambda \otimes x(k)$. The equation $A \otimes x(k) = \lambda \otimes x(k)$ are called problems of eigenvalue and eigenvectors or eigenproblems. Moreover, λ and $x(k)$ in equation $A \otimes x(k) = \lambda \otimes x(k)$ are called eigenvalue and eigenvector matrix A. ([8], [9], [10]) has also discussed the eigenproblem. Furthermore, [11] discussed the generalized eigenproblem in the max-plus algebra. The generalized eigenproblem is denoted as $A \otimes x = \lambda \otimes B \otimes x$ where $A,\ B \in \mathbb{R}_\varepsilon^{m\times n}$, $x \in \mathbb{R}_\varepsilon^n$, $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)$. Vector $x \in \mathbb{R}_\varepsilon^n$ and $\lambda \in \mathbb{R}_\varepsilon$ are eigenvector and eigenvalue, respectively.

Max-plus algebra has been generalized into interval max-plus algebra. Interval max-plus algebra is a set $I(\mathbb{R})_\varepsilon$ equipped with maximum ($\overline{\oplus}$) and plus ($\overline{\otimes}$) operations [12], [13]. Siswanto [14] has discussed the eigenvector space of matrix over interval max-plus algebra. Included in the discussion of eigenvector space is how to determine the basis of the eigenvector space and the dimensions of the space. Based on the results obtained by Cuninghame-Green and Butkovic on eigenproblems generalized in the max-plus algebra, [11] and [12]. This study will be discussed about the problem of eigenvalue and eigenvector (eigenproblem) that is generalized and the necessary and sufficient condition of eigenproblem that is common in the interval max-plus algebra can be solved.

2. RESEARCH METHODS

This research will discuss the concepts needed in the discussion before finding the results. These necessary concepts need to be well understood so that researchers can use them to bring up new definitions and theorems in interval max-plus. The research method used is to study the concepts in the max-plus algebra. Studies have been conducted on the definitions and theorems underlying this research.

2.1 Eigenproblem and Generalized Eigenproblem in Max-Plus Algebra

In this section is presented the definition of max-plus algebra and the matrix over max-plus algebra and its operations [7], [8]. Maximum ($\oplus$) and sum ($\otimes$) operations of a matrix over max-plus algebra, defined as follows,

a. For matrices $A = \left(A_{ij}\right)$, $B = \left(B_{ij}\right) \in \mathbb{R}_\varepsilon^{m\times n}$ defined by $A \oplus B \in \mathbb{R}_\varepsilon^{m\times n}$ with $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \max(A_{ij}, B_{ij})$.

b. For matrices $A = \left(A_{ij}\right) \in \mathbb{R}_\varepsilon^{m\times k}$, $B = \left(B_{ij}\right) \in \mathbb{R}_\varepsilon^{k\times n}$ be defined by $A \otimes B \in \mathbb{R}_\varepsilon^{m\times n}$ with $(A \otimes B)_{ij} = \bigoplus_{j=1}^{k} (A_{ij} \otimes B_{ij}) = \max_{j\in\{1,2,3,\dots,k\}} (A_{ij} + B_{ij})$.

Definition 1. [7], [8]. Given $A \in \mathbb{R}_\varepsilon^{n\times n}$, $\lambda \in \mathbb{R}_\varepsilon$ and $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)^T \in \mathbb{R}_\varepsilon^n$. The equation $A \otimes x = \lambda \otimes x$ is called eigenproblem. The set of all eigenvalue and all eigenvector of matrix A is denoted,

$$\Lambda(A) = \{\lambda \in \mathbb{R}_\varepsilon | \exists x \in \mathbb{R}_\varepsilon^n, x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)^T \ni A \otimes x = \lambda \otimes x\} \text{ and}$$
$$V(A) = \{x \in \mathbb{R}_\varepsilon^n - \{(\varepsilon, \varepsilon, \dots, \varepsilon)^T\} | A \otimes x = \lambda \otimes x, \lambda \in \mathbb{R}_\varepsilon\}.$$

Furthermore, given $A \in \mathbb{R}_\varepsilon^{m\times n}$, $\lambda \in \mathbb{R}_\varepsilon$ and $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)^T \in \mathbb{R}_\varepsilon^n$. The equation $x^T \otimes A = \lambda \otimes x^T$ is called left eigenproblem

Definition 2. [8], [11]. Given $A, B \in \mathbb{R}_\varepsilon^{m\times n}$, $\lambda \in \mathbb{R}_\varepsilon$ and $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)^T \in \mathbb{R}_\varepsilon^n$. The equation $A \otimes x = \lambda \otimes B \otimes x$ is called generalized eigenproblem. Given $A, B \in \mathbb{R}_\varepsilon^{m\times n}$, $\lambda \in \mathbb{R}_\varepsilon$ and $x \neq (\varepsilon, \varepsilon, \dots, \varepsilon)^T \in \mathbb{R}_\varepsilon^n$.

The equaion $x^T \otimes A = \lambda \otimes x^T \otimes B$ is called a left generalized eigenproblem. The set of all eigenvalue and eigenvector of matrices A and B is denoted,

$$\Lambda(A,B) = \{\lambda \in \mathbb{R}_\varepsilon | \exists x \in \mathbb{R}_\varepsilon^n, x \neq (\varepsilon,\varepsilon,\dots,\varepsilon)^T \ni A \otimes x = \lambda \otimes B \otimes x\} \text{ and}$$
$$V(A,B) = \{x \in \mathbb{R}_\varepsilon^n - \{(\varepsilon,\varepsilon,\dots,\varepsilon)^T\} | A \otimes x = \lambda \otimes B \otimes x, \lambda \in \mathbb{R}_\varepsilon\}.$$

Definition 3. [8], [11]. The generalized eigenproblem is called solvable if it has a solution and it is written (A,B) is solvable

2.2 Interval Max-Plus Algebra and Matrices over Interval Max-Plus Algebra

The closed interval x in $\mathbb{R}_\varepsilon$ is a subset of $\mathbb{R}_\varepsilon$ with $x =$ [lower bound, upper bound] $= [\underline{x}, \overline{x}] = \{x \in \mathbb{R}_\varepsilon | \underline{x} \leq x \leq \overline{x}\}$. The interval x in $\mathbb{R}_\varepsilon$ is called the max-plus interval. A number $x \in \mathbb{R}_\varepsilon$ can be expressed as an interval $[x, x]$.

Definition 4. [12], [14]. Let $I(\mathbb{R})_\varepsilon = \{x = [\underline{x}, \overline{x}] \,|\, \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \leq \overline{x}\} \cup \{\varepsilon\}$, where $\varepsilon = [\varepsilon, \varepsilon]$. On the set $I(\mathbb{R})_\varepsilon$ defined maximum $(\overline{\oplus})$ and plus $(\overline{\otimes})$ operations with $x \overline{\oplus} y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $x \overline{\otimes} y = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]$ for *all* $x, y \in I(\mathbb{R})_\varepsilon$. The set $I(\mathbb{R})_\varepsilon$ equipped with $\overline{\oplus}$ and $\overline{\otimes}$ operation is called interval max-plus algebra denoted as $I(\mathbb{R})_{max} = (I(\mathbb{R})_\varepsilon; \overline{\oplus}, \overline{\otimes})$. Neutral element and unit element are $\varepsilon = [\varepsilon, \varepsilon]$ and $\overline{0} = [0,0]$ respectively.

Example 5. Let $x = [2, 4]$ and $y = [-3, 5]$, so $x \overline{\oplus} y = [2 \oplus -3, 4 \oplus 5] = [2,5]$ and $x \overline{\otimes} y = [2 \otimes -3, 4 \otimes 5] = [-1,9]$.

Definition 6. [13], [14], [15]. The set of matrices in the size $m \times n$ with the elements in $I(\mathbb{R})_\varepsilon$ is denoted as $I(\mathbb{R})_\varepsilon^{m\times n}$, defined as $I(\mathbb{R})_\varepsilon^{m\times n} = \{\mathrm{A} = (\mathrm{A}_{ij}) \,|\mathrm{A}_{ij} \in I(\mathbb{R})_\varepsilon;\ i = 1,2,\dots,m,\ j = 1,2,\dots,n\}$.

Matrices that belong to $I(\mathbb{R})_\varepsilon^{m\times n}$ are known as matrices over interval max-plus algebra or interval matrices. Other definitions of matrices over interval max-plus algebra can be seen at [14].

Example 7. Let $A = \begin{bmatrix} [1,3] & [-2,4] \\ [-3,7] & [2,10] \end{bmatrix}$ and $B = \begin{bmatrix} [0,3] & [-4.6] \\ [-2.1] & [-6,-3] \\ [1.6] & [0.3] \end{bmatrix}$, A and B are matrices over interval max-plus algebra.

2.3 Eigenproblem of Matrices over Interval Max-Plus Algebra

Let $\mathrm{A} = (\mathrm{A}_{ij}) \in I(\mathbb{R})_\varepsilon^{n\times n}$, the eigenproblem of A is $\mathrm{A} \overline{\otimes} \mathrm{x} = \lambda \overline{\otimes} \mathrm{x}$. The vector $\mathrm{x} \in I(\mathbb{R})_\varepsilon^n$, $\mathrm{x} \neq (\varepsilon, \varepsilon, \dots, \varepsilon)^T$ and $\lambda \in I(\mathbb{R})_\varepsilon$ are eigenvector and eigenvalue of A.

Definition 8. [14]. Let $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{n\times n}$ where $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$ and $\lambda = [\underline{\lambda}, \overline{\lambda}] \in I(\mathbb{R})_\varepsilon$, defined

a. The set of the eigenvector of A corresponding to the eigenvalue λ is $\mathrm{V}(\mathrm{A},\lambda) = \{\mathrm{x} \in I(\mathbb{R})_\varepsilon^n \,|\mathrm{x} = [\underline{\mathrm{x}}, \overline{\mathrm{x}}] \ni \underline{\mathrm{x}} \in \mathrm{V}(\underline{\mathrm{A}}, \underline{\lambda});\ \overline{\mathrm{x}} \in \mathrm{V}(\overline{\mathrm{A}}, \overline{\lambda})\}$,

b. The set of the eigenvalue of A is

$$\Lambda(\mathrm{A}) = \{\lambda = [\underline{\lambda}, \overline{\lambda}] \in I(\mathbb{R})_\varepsilon |\ \mathrm{V}(\underline{\mathrm{A}}, \underline{\lambda}) \neq \{\mathcal{E}_{\mathrm{m}\times 1}\};\ \mathrm{V}(\overline{\mathrm{A}}, \overline{\lambda}) \neq \{\mathcal{E}_{\mathrm{m}\times 1}\}\},$$

c. $\mathrm{V}(\mathrm{A}) = \bigcup_{\lambda \in \Lambda(\mathrm{A})} \mathrm{V}(\mathrm{A}, \lambda)$,

d. $\mathrm{V}^+(\mathrm{A}, \lambda) = \mathrm{V}(\mathrm{A}, \lambda) \cap I(\mathbb{R})^{\mathrm{n}}$,

e. $\mathrm{V}^+(\mathrm{A}) = \mathrm{V}(\mathrm{A}) \cap I(\mathbb{R})^{\mathrm{n}}$.

3. RESULTS AND DISCUSSION

Given $\mathrm{A}, \mathrm{B} \in I(\mathbb{R})_\varepsilon^{m\times n}$, how to find $\mathrm{x} \in I(\mathbb{R})_\varepsilon^n$, $\mathrm{x} \neq (\varepsilon, \varepsilon, \dots, \varepsilon)$ and $\lambda \in I(\mathbb{R})_\varepsilon$ such that $\mathrm{A} \overline{\otimes} \mathrm{x} = \lambda \overline{\otimes} \mathrm{B} \overline{\otimes} \mathrm{x}$ is called the generalized eigenproblem in interval max-plus algebra. Furthermore, given $\mathrm{A}, \mathrm{B} \in I(\mathbb{R})_\varepsilon^{m\times n}$, how to find $\mathrm{x} \in I(\mathbb{R})_\varepsilon^n$, $\mathrm{x} \neq (\varepsilon, \varepsilon, \dots, \varepsilon)$ and $\lambda \in I(\mathbb{R})_\varepsilon$ such that $\mathrm{x}^T \overline{\otimes} \mathrm{A} = \lambda \overline{\otimes} \mathrm{x}^T \overline{\otimes} \mathrm{B}$ is called the left generalized eigenproblem in interval max-plus algebra.

Definition 1. Let $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ where $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{m\times n})_b$ and $\lambda = [\underline{\lambda}, \overline{\lambda}] \in I(\mathbb{R})_\varepsilon$, defined

a. The set of the eigenvalue is

$$\Lambda(A,B) = \{\lambda = [\underline{\lambda}, \overline{\lambda}] \in I(\mathbb{R})_\varepsilon \mid \underline{\lambda} \in \Lambda(\underline{A}, \underline{B});\ \overline{\lambda} \in \Lambda(\overline{A}, \overline{B})\}.$$

b. The set of the eigenvector corresponding to the eigenvalue λ is

$$V(A,B) = \{x \in I(\mathbb{R})_\varepsilon^n \mid x = [\underline{x}, \overline{x}] \ni \underline{x} \in V(\underline{A}, \underline{B});\ \overline{x} \in V(\overline{A}, \overline{B})\}.$$

If the generalized eigenproblem $A \overline{\otimes} x = \lambda \overline{\otimes} B \overline{\otimes} x$ has a solution then the generalized eigenproblem is called solvable. Furthermore, (A, B) is solvable. The following terms will be presented eigenvalue problem can be solved, where $A = (a_{ij})$, $B = (b_{ij}) \in I(\mathbb{R})^{m\times n}$ finite matrices. Let $M = \{1, 2, \dots, m\}$, $N = \{1, 2, \dots, n\}$, the matrices $C = (c_{ij}) = (a_{ij} \overline{\otimes} b_{ij}^{-1})$ and $D = (d_{ij}) = (b_{ij} \overline{\otimes} a_{ij}^{-1})$.

Theorem 2. *If (A, B) can be solved and $\lambda = [\underline{\lambda}, \overline{\lambda}] \in \Lambda(A, B)$ then C satisfies,*

a. $max_{i\in M} min_{j\in N} c_{ij} \le \underline{\lambda} \le min_{i\in M} max_{j\in N} c_{ij}$ and

b. $max_{i\in M} min_{j\in N} c_{ij} \le \overline{\lambda} \le min_{i\in M} max_{j\in N} c_{ij}$

Proof. Let $A \approx [\underline{A}, \overline{A}]$ and $B \approx [\underline{B}, \overline{B}]$. Noted that (A, B) is solvable so eigenproblem $A \overline{\otimes} x = \lambda \overline{\otimes} B \overline{\otimes} x$ has a solution. Since $A \overline{\otimes} x = \lambda \overline{\otimes} B \overline{\otimes} x$, so $A \overline{\otimes} x \approx [\underline{A} \oplus \underline{x}, \overline{A} \oplus \overline{x}]$ and $\lambda \overline{\otimes} B \overline{\otimes} x \approx [\underline{\lambda} \otimes \underline{B} \otimes \underline{x}, \overline{\lambda} \otimes \overline{B} \otimes \overline{x}]$. As a result, $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{B} \otimes \underline{x}$ and $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{B} \otimes \overline{x}$ have a solution. Therefore, $(\underline{A}, \underline{B})$ and $(\overline{A}, \overline{B})$ are solvable, so $max_{i\in M}\, min_{j\in N} c_{ij} \le \underline{\lambda} \le min_{i\in M}\, max_{j\in N} c_{ij}$ and $max_{i\in M}\, min_{j\in N} c_{ij} \le \overline{\lambda} \le min_{i\in M}\, max_{j\in N} c_{ij}$. ■

Corollary 3. *If (A, B) is solvable then C satisfies,*

a. $max_{i\in M} min_{j\in N} c_{ij} \le min_{i\in M} max_{j\in N} c_{ij}$ and

b. $max_{i\in M} min_{j\in N} c_{ij} \le min_{i\in M} max_{j\in N} c_{ij}$

Proof. Noted that (A, B) is solvable and $\lambda = [\underline{\lambda}, \overline{\lambda}] \in \Lambda(A, B)$. According to Theorem 2, we obtain $max_{i\in M} min_{j\in N} c_{ij} \le \underline{\lambda} \le min_{i\in M} max_{j\in N} c_{ij}$ and $max_{i\in M} min_{j\in N} c_{ij} \le \overline{\lambda} \le min_{i\in M} max_{j\in N} c_{ij}$ Therefore, $max_{i\in M} min_{j\in N} c_{ij} \le min_{i\in M} max_{j\in N} c_{ij}$ and $max_{i\in M} min_{j\in N} c_{ij} \le min_{i\in M} max_{j\in N} c_{ij}$. ■

Corollary 4. *If $m = n$, (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then C hold $\lambda'(C) \le \lambda(C)$.*

Proof. Noted that $m = n$, (A, B) solvable and $\lambda \in \Lambda(A, B)$. Therefore, $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{B} \otimes \underline{x}$ dan $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{B} \otimes \overline{x}$ have solution. Besides that, $\underline{\lambda} \in \Lambda(\underline{A}, \underline{B})$ *and* $\overline{\lambda} \in \Lambda(\overline{A}, \overline{B})$, $\underline{\lambda}'(\underline{C}) \le \underline{\lambda}(\underline{C})$ and $\overline{\lambda}'(\overline{C}) \le \overline{\lambda}(\overline{C})$. Therefore, $\lambda'(C) \le \lambda(C)$. ■

Corollary 5. *If (A, B) is solvable then the greatest diagonal element* of $A \overline{\otimes}' B^*$ *less than the smallest of the diagonal element* $A \overline{\otimes} B^*$.

Proof. Let $A \approx [\underline{A}, \overline{A}]$ and $B \approx [\underline{B}, \overline{B}]$. Noted that, (A, B) is solvable so eigenproblem $A \overline{\otimes} x = \lambda \overline{\otimes} B \overline{\otimes} x$ has a solution. Since $A \overline{\otimes} x = \lambda \overline{\otimes} B \overline{\otimes} x$, so $A \overline{\otimes} x \approx [\underline{A} \oplus \underline{x}, \overline{A} \oplus \overline{x}]$ and $\lambda \overline{\otimes} B \overline{\otimes} x \approx [\underline{\lambda} \otimes \underline{B} \otimes \underline{x}, \overline{\lambda} \otimes \overline{B} \otimes \overline{x}]$. As a result, $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{B} \otimes \underline{x}$ and $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{B} \otimes \overline{x}$ have a solution. Therefore, $(\underline{A}, \underline{B})$ and $(\overline{A}, \overline{B})$ are solvable. As a result, the greatest diagonal element of $\underline{A} \otimes' \underline{B}^*$ is less than the smallest of the diagonal element $\underline{A} \otimes \underline{B}^*$. As well, less than the smallest of the diagonal element $\overline{A} \otimes' \overline{B}^*$ less than the smallest of the diagonal element $\overline{A} \otimes \overline{B}^*$. Therefore, the greatest diagonal element of $A \overline{\otimes}' B^*$ is less than the smallest of the diagonal element $A \overline{\otimes} B^*$. ■

Corollary 6. *If (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then* $\lambda'(A \overline{\otimes}' B^*) \le \lambda \le \lambda(A \overline{\otimes} B^*)$.

Proof. Let $A \approx [\underline{A}, \overline{A}]$ and $B \approx [\underline{B}, \overline{B}]$. Noted that, (A, B)solvable so the eigenproblem $A \overline{\otimes} x = \lambda \overline{\otimes} B \overline{\otimes} x$ has a solution. Since $A \overline{\otimes} x = \lambda \overline{\otimes} B \overline{\otimes} x$, so $A \overline{\otimes} x \approx [\underline{A} \oplus \underline{x}, \overline{A} \oplus \overline{x}]$ and $\lambda \overline{\otimes} B \overline{\otimes} x \approx [\underline{\lambda} \otimes \underline{B} \otimes \underline{x}, \overline{\lambda} \otimes \overline{B} \otimes \overline{x}]$. As a result, $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{B} \otimes \underline{x}$ and $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{B} \otimes \overline{x}$ have a solution. Therefore, $(\underline{A}, \underline{B})$ and $(\overline{A}, \overline{B})$ solvable. Noted that $\underline{\lambda} \in \Lambda(\underline{A}, \underline{B})$ and $\overline{\lambda} \in \Lambda(\overline{A}, \overline{B})$. As a result,

$\underline{\lambda}'(\underline{A} \otimes' \underline{B}^*) \leq \underline{\lambda} \leq \underline{\lambda}(\underline{A} \otimes \underline{B}^*)$ and $\overline{\lambda}'(\overline{A} \otimes' \overline{B}^*) \leq \overline{\lambda} \leq \overline{\lambda}(\overline{A} \otimes' \overline{B}^*)$. Therefore, $\lambda'(A \overline{\otimes}' B^*) \leq \lambda \leq \lambda(A \overline{\otimes} B^*)$. ■

Furthermore, let $D_i = \begin{pmatrix} A_{i1} & \dots & A_{in} \\ B_{i1} & \dots & B_{in} \end{pmatrix}$, $i = 1, 2, 3, \dots .\ m.$

Theorem 7. *The* (A, B) *can be solved if only if the column space of* D_i *have a common element so that the equal multiples states depend on for all i, that there is* $\xi = (\xi_1, \xi_2)^T$ *and* $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_n)^T$*s* $\mathrm{D}_i \overline{\otimes}\ \mathrm{x} = \mathrm{y}_i \overline{\otimes}\ \xi$ for *some* $\mathrm{y}_i \in I(\mathbb{R})$ for *all* $i = 1, 2, \dots, m$. Therefore, $\xi_1 \otimes\ \xi_2^{-1} \in \Lambda(\mathrm{A}, \mathrm{B})$.

Proof. The (A, B) to be solvable so $(\underline{A}, \underline{B})$ and $(\overline{A}, \overline{B})$ to be solvable. As a result, the column space of having an element of $\underline{D}_i$ the common elements so that the multiples of the same states depend on all i, that there is $\underline{\xi} = \left(\underline{\xi_1}, \underline{\xi_2}\right)^T$ and $\underline{x} = (\underline{x_1}, \underline{x_2}, \dots, \underline{x_n})^T$such that $\underline{D_i} \otimes \underline{x} = \underline{y_i} \otimes \underline{\xi}$ some $\underline{y_i} \in \mathbb{R}$ all $i = 1, 2, \dots, m$. If this case then $\underline{\xi_1} \otimes\ \underline{\xi_2}^{-1} \in \Lambda(\underline{A}, \underline{B})$. As well, the column space of having an element of $\overline{\mathrm{D}}_i$ the common elements so that the multiples of the same states depend on all i, that there is $\overline{\xi} = \left(\overline{\xi}_1, \overline{\xi}_2\right)^T$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n)^T$such that $\overline{D}_i \otimes \overline{x} = \overline{y}_i \otimes \overline{\xi}$ some $\overline{y}_i \in \mathbb{R}$ all $i = 1, 2, \dots, m$. If this case then $\overline{\xi}_1 \otimes\ \overline{\xi}_2^{-1} \in \Lambda(\overline{A}, \overline{B})$. Therefore, the column space of having an element of D_i the common elements so that the multiples of the same states depend on all i, that there is $\xi = (\xi_1, \xi_2)^T$ and $x = (x_1, x_2, \dots, x_n)^T$such that $D_i \overline{\otimes}\ x = y_i \overline{\otimes}\ \xi$ some $y_i \in I(\mathbb{R})$ all $i = 1, 2, \dots, m$. If this case then $\xi_1 \otimes\ \xi_2^{-1} \in \Lambda(A, B)$. Otherwise, the column space of having an element of D_i the common elements so that the multiples of the same states depend on all i, that there is $\xi = (\xi_1, \xi_2)^T$ and $x = (x_1, x_2, \dots, x_n)^T$such that $D_i \overline{\otimes}\ x = y_i \overline{\otimes}\ \xi$ some $\mathrm{y}_i \in I(\mathbb{R})$ all $i = 1, 2, \dots, m$. If this case then $\xi_1 \otimes\ \xi_2^{-1} \in \Lambda(A, B)$. As a result,

a. the column space of having an element of $\underline{\mathrm{D}}_\mathrm{i}$ the common elements so that the multiples of the same states depend on all i, that there is $\underline{\xi} = \left(\underline{\xi_1}, \underline{\xi_2}\right)^T$ and $\underline{x} = (\underline{x_1}, \underline{x_2}, \dots, \underline{x_n})^T$such that $\underline{D_i} \otimes \underline{x} = \underline{y_i} \otimes \underline{\xi}$ some $\underline{y_i} \in \mathbb{R}$ all $i = 1, 2, \dots, m$. If this case then $\underline{\xi_1} \otimes\ \underline{\xi_2}^{-1} \in \Lambda(\underline{A}, \underline{B})$. Therefore, $(\underline{A}, \underline{B})$ to be sovable.

b. the column space of having an element of $\overline{\mathrm{D}}_i$ the common elements so that the multiples of the same states depend on all i, that there is $\overline{\xi} = \left(\overline{\xi}_1, \overline{\xi}_2\right)^T$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n)^T$such that $\overline{D}_i \otimes \overline{x} = \overline{y}_i \otimes \overline{\xi}$ some $\overline{y}_i \in \mathbb{R}$ all $i = 1, 2, \dots, m$. If this case then $\overline{\xi}_1 \otimes\ \overline{\xi}_2^{-1} \in \Lambda(\overline{A}, \overline{B})$. Therefore, $(\overline{A}, \overline{B})$ to be solvable.

From a and b, we can conclude (A, B) to be solvable. ■

Furthermore, we will study the uniqueness of eigenvalue. Look again, the generalized left eigenproblem in interval max-plus algebra and the generalized right eigenproblem in interval max-plus algebra.

Theorem 8. *If the generalized eigenproblem and the left generalized eigenproblem for* (A, B) are *solvable then both have an unique and identic eigenvalue, that* $\Lambda(A, B) = \Lambda(A^T, B^T) = \{\lambda\}$ and $\Lambda(A, B) = \emptyset$, $\Lambda(A^T, B^T) = \emptyset$.

Proof. Let $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$ and $B \approx [\underline{B}, \overline{B}]$. Note that, (A, B) solvable so the eigenproblem $\mathrm{A} \overline{\otimes}\ \mathrm{x} = \lambda \overline{\otimes}\ \mathrm{B} \overline{\otimes}\ \mathrm{x}$ has a solution. Because $A \overline{\otimes}\ x = \lambda \overline{\otimes}\ B \overline{\otimes}\ x$, so $A \overline{\otimes}\ x \approx [\underline{A} \oplus \underline{x}, \overline{A} \oplus \overline{x}]$ and $\lambda \overline{\otimes}\ B \overline{\otimes}\ x \approx [\underline{\lambda} \otimes \underline{B} \otimes \underline{x}, \overline{\lambda} \otimes \overline{B} \otimes \overline{x}]$. As a result, $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{B} \otimes \underline{x}$ and $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{B} \otimes \overline{x}$ has a solution, both have eigenvalue unique and identic, that

1. $\Lambda(\underline{A}, \underline{B}) = \Lambda(\underline{A}^T, \underline{B}^T) = \{\underline{\lambda}\}$ *and* $\Lambda(\underline{A}, \underline{B}) = \emptyset$, $\Lambda(\underline{A}^T, \underline{B}^T) = \emptyset$.
2. $\Lambda(\overline{A}, \overline{B}) = \Lambda\left(\overline{A}^T, \overline{B}^T\right) = \{\overline{\lambda}\}$ *and* $\Lambda(\overline{A}, \overline{B}) = \emptyset$, $\Lambda\left(\overline{A}^T, \overline{B}^T\right) = \emptyset$.

Therefore, $\Lambda(A, B) = \Lambda(A^T, B^T) = \{\lambda\}$ and $\Lambda(A, B) = \emptyset$, $\Lambda(A^T, B^T) = \emptyset$. ■

Corollary 9. *If* $\mathrm{A}, \mathrm{B} \in I(\mathbb{R})_\varepsilon^{n \times n}$ *are symmetric then* $|\Lambda(A, B)| \leq 1$.

Proof. If $\mathrm{A}, \mathrm{B} \in I(\mathbb{R})_\varepsilon^{n \times n}$ are symmetric, so $\underline{A}, \underline{B}, \overline{A}, \overline{B} \in \mathbb{R}_\varepsilon^{n \times n}$, $|\Lambda(\underline{A}, \underline{B})| \leq 1$ and $|\Lambda(\overline{A}, \overline{B})| \leq 1$. Therefore, $|\Lambda(A, B)| \leq 1$. ■

4. CONCLUSION

Based on the description in the discussion can be concluded as follows:

a. One definition, three theorems and five corollaries are obtained in the section 3 that discuss the existence of the solution to generalized eigenproblem in interval max-plus algebra.

b. The important results presented in **Theorem 7** and **Theorem 8**.
The **Theorem 7** states that the (A, B) can be solved if only if the columns space of D_i have a common element so that the equal multiples states depend on for all *i*. The **Theorem 8** states if the generalized eigenproblem and the left generalized eigenproblem for (A, B) are solvable then both the generalized eigenproblem and the left generalized eigenproblem have an unique and identic eigenvalue.

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