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SOME PROPERTIES ON COPRIME GRAPH OF GENERALIZED QUATERNION GROUPS

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ABSTRACT

A coprime graph is a representation of finite groups on graphs by defining the vertex graph as an element in a group and two vertices adjacent to each other's if and only if the order of the two elements is coprime. In this research, we discuss the generalized Quaternion group and its properties. Then we discuss the properties of the coprime graph over the generalized Quaternion group by looking at its Eulerian, Hamiltonian, and Planarity sides. In general, the coprime graphs of the generalized quaternion group are not Eulerian, not Hamilton, and not planar graphs. The coprime graph of the generalized quaternion group Q_{4n} is a planar graph if $n = 2^k$ for a natural number k.

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1. INTRODUCTION

The quaternion group was first introduced by Sir William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional space [1]. Hamilton uses this group in both theoretical and applied mathematics, particularly for calculus, including three-dimensional rotations. Furthermore, the quaternion group is widely applied in various fields. In Physics, for example, this group has many roles, mainly in the field of quantum mechanics, as stated in [2]. The quaternion group also holds an important role in the field of computer graphics, primarily for 3D rotation transformations, as written in [3]. In addition, this group also has a role in the fields of aerospace, orbits, and virtual reality, as research has been done in [4].

One of the methods to learn about finite groups is to represent them on a graph. The history of representations of finite groups on graphs begins with the definition of the Cayley graph by Artur Cayley (1878) [5]. Artur Cayley represents the finite group G and a subset $A \subset G$ (called the generator of Cayley graph) by defining the elements of the group as vertices and adjacency of any vertex g and h hold if only if $g = ah$, with $a \in A$. Further research on the Cayley graph was carried out by replacing the objects, finite groups replaced by finite semigroups, and it was discussed in many research, for example [6], [7], and [8]. Cayley graphs can also be applied to form cryptographic systems, as researched in [9].

The follow-up research on the representation of finite groups in graphs that give much attention to mathematical researchers is the power graph defined by Kalarev and Quinn [10]. They define the adjacency of two vertices g and h, on the power graph if there is a natural number n such that $g = h^n$. The power graph was introduced by [10] is a representation of semigroups which are directed graphs, while the undirected version was discussed later by [11]. Discussions about power graphs on finite groups are done by [12], [13], and [14]. At the same time, the research on power graphs of torsion-free groups is discussed in [15].

The following research on the representation of finite groups in graphs that also received attention is the coprime graph. Coprime graph defined on finite groups G by viewing the elements in the group as vertexes, and the elements $g, h \in G$ are connected if the orders of g and h are coprime. Research on coprime graphs in cyclic and dihedral groups was done by [16] and [17], while [18] discussed coprime graphs in Z_n groups and their subgroups. Research from [19] discusses the coprime graph in the generalized quaternion group. In this research, we will complete the results of [19] by looking at the coprime graph of the generalized quaternion group from the Eulerian, Hamiltonian, and Planarity sides.

2. RESEARCH METHODS

The research method used is a literature study. The detailed steps in conducting research are as follows:

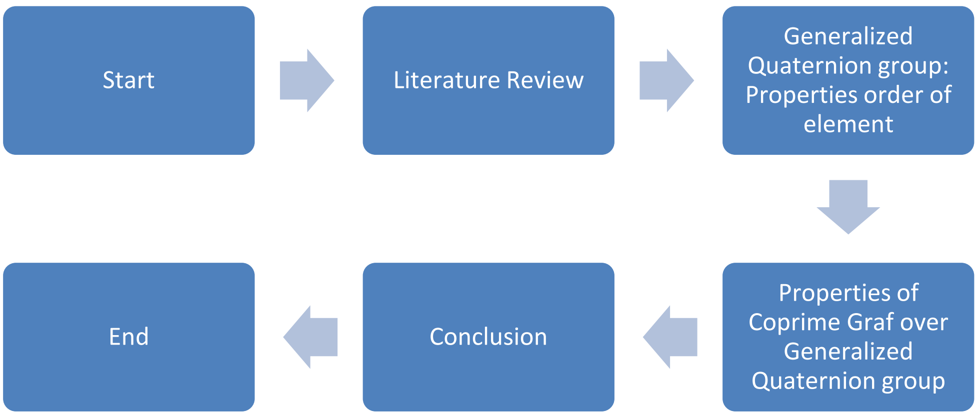

Figure 1. Research method

We study articles relating to the representation of finite groups on graphs, especially on coprime graphs and articles on generalized quaternion groups. In particular, the study began by looking for the properties of generalized quaternion groups, especially those relating to the order of elements. Furthermore, we look at the properties of coprime graphs of generalized quaternion groups using the properties of generalized quaternion groups and looking at the specific properties of the quaternion group in certain orders.

Terminology about graphs is taken from sources [20], [21], or [22], while terminology about finite groups theory is taken from sources [23].

Theorem 1 [22] *If Graph G contains an odd cycle, then G is not a bipartite graph.*

Theorem 2. [20] *The connected graph G is Eulerian if every vertex in G has an even degree.*

Theorem 3. [21] *Given graph G. If G is Hamilton's graph, then for any $\emptyset \neq S \subseteq V(G)$, the number of components of $G - S$ is less than the number of members of vertices is set S.*

Theorem 4. [20] *A graph is a planar graph if and only if it does not contain a subgraph that is homeomorphic with K_5 or $K_{3,3}$.*

Definition 5. [21] A vertex cut of a connected graph G is a subset of the vertex set $S \subseteq V(G)$ such that $G - S$ has more than one connected component. In other words, a vertex cut is a subset of vertices of a connected graph that disconnects the graph if removed together with any incident edges.

Definition 6. [21] A graph G on more than two vertices is said to be k-connected if there does not exist a vertex cut of size $k-1$ whose removal disconnects the graph.

3. RESULTS AND DISCUSSION

This section begins with a discussion of the properties of generalized quaternion groups. In this section two properties are found relating to the generalized quaternion group. The first theorem relates to the general form of the elements in the group, while the next corresponds to the order of each element of the group. These two properties are an important part of the next discussion.

3.1 Generalized Quaternion Group

The discussion begins with the definition of a generalized quaternion group as follows

Definition 7. [9] A generalized quaternion group (Q_{4n}) is a group whose membership is defined as follows

$$Q_{4n} = \{a, b | a^{2n} = b^4 = e, b^{-1}ab = a^{-1}\}.$$

Here are some of the properties of the generalized quaternion group used in the next discussion.

Theorem 8. [24] *Let Q_{4n} be a generalized quaternion group. Then*

1. *Q_{4n} abelian if and only if $n = 1$*
2. *Every element in Q_{4n} can be written uniquely as $a^i b^j$ where $0 \leq i < 2n$ and $b = 0,1$.*
3. *$|Q_{4n}| = 4n$*

Proof. Suppose Q_{4n} is a generalized quaternion group. Then

1. If $n = 1$, then $a = b^2$ which means $Q_4 \cong Z_4$ thus Q_4 is an abelian group.
2. Based on the definition of a generalized Quaternion group, elements in that group will be $a^i b^j$, since $a^{2n} = e$ and $b^4 = e$ are for $0 \leq i < 2n$ and $0 \leq j < 4$, then $a^n = b^2$. Thus $a^i b^2 = a^i a^n = a^{i+n}$, while $a^i b^3 = a^i b^2\, b = a^i a^n\, b = a^{i+n}\, b$. So the limit for j can be changed to $j = 0,1$.
3. As a result of the second statement, then the element contained in the Q_{4n} is $4n$. ■

Since this research is based on the order elements of the quaternion group, so the following theorem plays an important role in the next discussion. This theorem has been written in [19], in this study we rewrite it with a different proof.

Theorem 9. *Suppose that $G = Q_{4n}$ is a generalized quaternion group. The order of each element in Q_{4n}*

$$o(a^i b^j) = \begin{cases} \dfrac{2n}{\gcd(i, 2n)}, & j = 0 \\ 4\quad, & j = 1 \end{cases}$$

Proof. Let $g = a^i \in Q_{4n}$ where $0 \le i < 2n$. Note that for any natural number $n, \gcd(i, 2n) \mid i$. If $\gcd(i, 2n) = l$, then $i = kl$ for a natural number k. Then

$$\left(a^i\right)^{\frac{2n}{\gcd(i,2n)}} = \left(a^i\right)^{\frac{2n}{l}} = a^{\frac{2i}{l}n} = a^{2kn} = (a^n)^{2k} = e.$$

Its mean order of any $a^i \in Q_{4n}$ is $\frac{2n}{\gcd(i,2n)}$.

Furthermore, in the case of $j = 1$, then the element in the Q_{4n} will be $a^i b$ with $1 \le i \le 2n$. Since $b^{-1}\,ab = a^{-1}$, then $ab = ba^{-1}$, so the elaboration of $\left(a^i b\right)^2$ gives the following results

$$\begin{aligned}\left(a^i b\right)^2 &= \left(a^i b\right)\left(a^i b\right)\\ &= a^{i-1} b a^{-1} a a^{i-1} b\\ &= a^{i-1} b a^{i-1} b\\ &= a^{i-1} b a^{i-1} b\\ &= \left(a^{i-2} b a^{-1}\right)\left(a a^{i-2} b\right)\\ &= a^{i-2} b a^{i-2} b.\end{aligned}$$

By continuing the process iteratively, the final equation will be in the form $(ab)(ab) = ba^{-1}\,ab = b^2$. Its mean that $\left(a^i b\right)^2 = b^2$ for any $0 \le i < 2n$, so

$$\left(a^i b\right)^4 = \left(a^i b\right)^2 \left(a^i b\right)^2 = (b^2)(b^2) = b^4 = e. \blacksquare$$

This section will discuss the coprime graph on the generalized quaternion group. We investigate the properties of the graph that appear form coprime graph on the generalized quaternion group (Q_{4n}) for any natural number n of the. Then, based on the selected value of n, we will see Hamiltonian, Eulerian, and planarity of the graph formed.

3.2 Coprime Graph of Generalized Quaternion Group

The discussion in this section begins with the definition of a coprime graph in any group.

Definition 10. [16] Given a finite group G. Coprime graph of G (noted as Γ_G) is a graph with a vertex set is elements in group G and two vertexes $g, h \in V(\Gamma_G)$ adjacent if and only if $\gcd\left(o(g), o(h)\right) = 1$.

Based on the definition above, vertex g and h are adjacent if and only if $\gcd\left(o(g), o(h)\right) = 1$. Since $\gcd(a, b) = \gcd(b, a)$, so the coprime graph is an undirected graph.

Example 11**.** Coprime graph of group quaternion Q_8 and Q_{12} are as follows

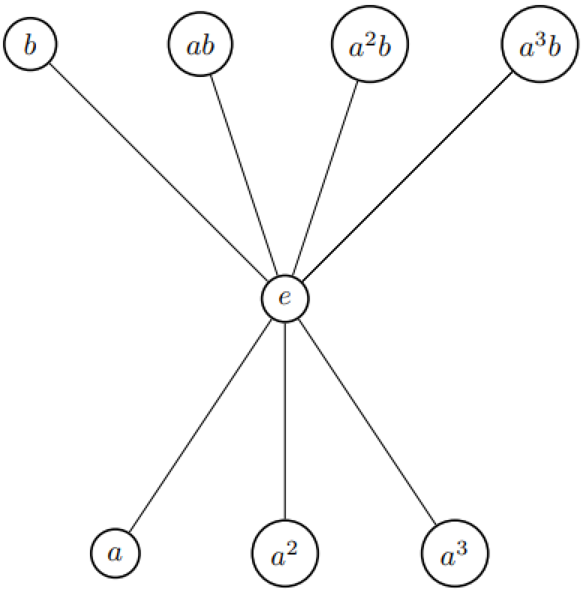

Figure 2. Coprime Graph of Q_8 (Γ_{Q_8})

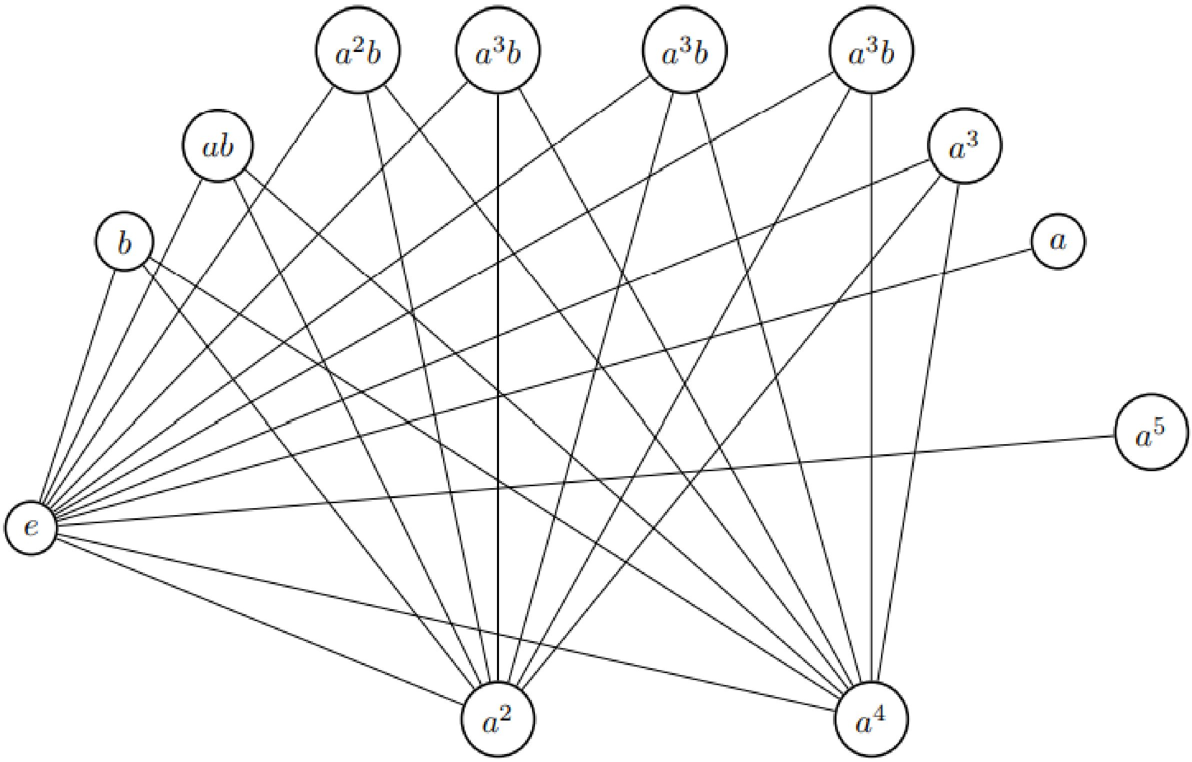

Figure 3. Coprime graph of Q_{12} ($\Gamma_{Q_{12}}$)

Theorem 12. *Given $\Gamma_{Q_{4n}}$, the coprime graph of Q_{4n} with n is odd numbers. Then*

1. $der_{\Gamma_{Q_{4n}}}(a^i b) = n$, *for any* $0 \leq i < 2n$.
2. $der_{\Gamma_{Q_{4n}}}(a^1) = der_{\Gamma_{Q_{4n}}}(a^{2n-1}) = 1$
3. $\Gamma_{Q_{4n}}$ *not Hamiltonian*
4. $\Gamma_{Q_{4n}}$ *not Eulerian*
5. $\Gamma_{Q_{4n}}$ *not Planar graph*
6. $\Gamma_{Q_{4n}}$ *is* $1-$*connected*

Proof. Suppose $\Gamma_{Q_{4n}}$ is the coprime graph of Q_{4n} with n is odd numbers. Then,

1. Based on Theorem 9, $o(a^i b) = 4$ for any $0 \leq i \leq 2n$. Since $o(a^j) = \frac{2n}{\gcd(j,2n)}$ for any $j = 2k, 1 \leq k \leq n$, then
$$o(a^j) = o(a^{2k}) = \frac{2n}{\gcd(2k,2n)} = \frac{2n}{2} = n.$$
As n be odds numbers, then $\gcd\left(o(a^{2k}), o(a^i b)\right) = 1$. So every vertex on $\{a^{2k} | 1 \leq k \leq n\}$ adjacent with every vertex on $\{a^i b | 1 \leq i \leq 2n\}$. It means $der_{\Gamma_{Q_{4n}}}(a^i b) = n$, for any $0 \leq i < 2n$.
2. It clearly understands $o(a) = o(a^{2n-1}) = 2n$. As a consequence of Theorem 9, the order of any element in Q_{4n} is a factor of $2n$ or 4. So vertex a and a^{2n-1} are only connected with the identity element on Q_{4n}. Its mean $der_{\Gamma_{Q_{4n}}}(a^1) = der_{\Gamma_{Q_{4n}}}(a^{2n-1}) = 1$
3. As a consequence of the second statement, $\Gamma_{Q_{4n}}$ not Hamiltonian.
4. As a result of the first statement and consequence of n is an odd number than $\Gamma_{Q_{4n}}$, not Eulerian.
5. Based on theorem 3, $o(a^i b) = 4$ for any $0 \leq i < 2n$ and $o(a^{2k}) = n$ for any $1 \leq k < n$. For $n = 3$, $o(e) = 1$, $o(a^2) = 3$ and $o(a^4) = 3$, so vertexes e, a^2, a^3 and $ab, a^2, a^3 b$ form a subgraf $K_{3,3}$ of $\Gamma_{Q_{12}}$. So $\Gamma_{Q_{12}}$ not a planar graph. Furthermore for $n > 3$, $o(a^2) = (a^4) = \cdots = o(a^{2(n-1)}) = n$ so that the vertek set $\{a^2, a^4, \ldots, a^{2(n-1)}\}$ is entirely connected to the vertek set $\{a^i b | 0 \leq i < 2n\}$. So $\Gamma_{Q_{4n}}$ with odd numbers $n > 3$ contains a subgraph $K_{3,3}$ which means not a planar graph.
6. Based on the second statement, coprime graph $\Gamma_{Q_{4n}}$ contains end-vertex. So $\Gamma_{Q_{4n}}$ is a 1-connected. ■

Theorem 13. *If $\Gamma_{Q_{4n}}$ is a coprime graph of Q_{4n} with n odd primes, then $\Gamma_{Q_{4n}}$ is a tripartite graph.*

Proof. We begin the proof by forming a partition of the element in Q_{4n} as follows $A = \{e\}$, $B = \{a^{2k} | 1 \leq k < n\}$ and $C = \{a^i b, a^{2k+1} | 1 \leq i < 2n, 1 \leq k < n\}$. Note that the order of each element from each partition is as follows $o(e) = 1$, $o(a^{2k}) = n$ and $o(a^{2k+1}) = 2n$ for $1 \leq k \leq n$, while $o(a^i b) = 4$ for $1 \leq i < 2n$. Since n is an odd prime, then the vertex in each set of partitions A, B, or C cannot be mutually adjacent. At the same time, the vertex in A is adjacent with all vertices in B and C and the vertex in set B is an adjacent with the vertex in set C (except vertexes in the form a^{2k+1}). In other words, $\Gamma_{Q_{4n}}$ is a tripartite graph but not a complete tripartite graph. ■

Theorem 14. *If $\Gamma_{Q_{4n}}$ is a coprime graph of Q_{4n} with $n = 2^k$ for a natural number k, then $\Gamma_{Q_{4n}}$ is a star graph S_{4n}.*

Proof. Since $n = 2^k$, then based on Theorem 9, the order of any element in the group Q_{4n} is a factor of 2^{k+1} for a natural number k**.** It mean that for any $g, h \in Q_{4n}$ with $g, h \neq e$, then the order of g and h will not be coprime, which means g and h are not adjacent in coprime graph $\Gamma_{Q_{4n}}$. Because the identity element is always adjacent to all the elements in the group, then $\Gamma_{Q_{4n}}$ is star graph S_{4n}. ■

Theorem 15. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{km}$, where p_i is an odd prime number and k_i is a non-negative integer for every i=1,2,...,m, then*

1. *Graph $\Gamma_{Q_{4n}}$ has a vertex of degree one*
2. *Graph $\Gamma_{Q_{4n}}$ is not a planar graph.*

Proof. Suppose $\Gamma_{Q_{4n}}$ is a coprime graph of a generalized quaternion group Q_{4n} with $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{km}$. Then

1. Since $o(a) = o(a^{2n-1}) = 2n = 2p_1^{k_1} p_2^{k_2} \ldots p_m^{km}$, thus based on Theorem 9, the order of other elements will not be coprime with the order of a. So vertex a and a^{2n-1} are only connected with vertex identity e. In other words, vertex a and a^{2n-1} have a degree of one or become the end vertex of the graph $\Gamma_{Q_{4n}}$.
2. The proof of this statement is in line with the proof of **Theorem 12** number 5. ■

Collorary 16. *If $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{km}$, where p_i is an odd prime number and k_i is a non-negative integer for every i=1,2, ..., m, then*

1. *Graph $\Gamma_{Q_{4n}}$ is not Eulerian*
2. *Graph $\Gamma_{Q_{4n}}$ is not Hamiltonian.*

Proof. Suppose $\Gamma_{Q_{4n}}$ is a coprime graph of a generalized quaternion group Q_{4n} with $n = p_1^{k_1} p_2^{k_2} \ldots p_m^{km}$. Then

1. Since vertexes a and a^{2n-1} have degrees one, so $\Gamma_{Q_{4n}}$ contains vertex with odd degrees. It means $\Gamma_{Q_{4n}}$ is not an Eulerian graph.
2. Note that vertex e is connected to all vertices in the graph $\Gamma_{Q_{4n}}$. Based on Theorem 15, vertex a and a^{2n-1} have degrees one. Chosen $S = \{e\}$, then $\Gamma_{Q_{4n}} - S$ has three components. Thus the components of $\Gamma_{Q_{4n}} - S$ are more than $|S|$ for any n. Based on Theorem 3, then $\Gamma_{Q_{4n}}$ is not Hamilton's graph. ■

The following corollary is a result of **Theorem 14** and **Theorem 15**.

Corollary 17. *If $\Gamma_{Q_{4n}}$ is the coprime graph of Q_{4n}, then $\Gamma_{Q_{4n}}$ is a planar graph if and only if $n = 2^k$ for a natural number k.*

Corollary 18. *If $\Gamma_{Q_{4n}}$ is the coprime graph of Q_{4n}, then $\Gamma_{Q_{4n}}$ is not Eulerian and not Hamiltonian for any natural number n.*

4. CONCLUSIONS

For any natural number n, the coprime graph of the generalized quaternion group is not a Hamilton graph and not an Euler graph. This is because for any n, it can always be found vertek with degree one in $\Gamma_{Q_{4n}}$. Furthermore, graph $\Gamma_{Q_{4n}}$ is a graph of stars if $n = 2^k$ for a natural number k. So, $\Gamma_{Q_{4n}}$ is a planar graph if only if $n = 2^k$ for a natural number k.

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