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ON PROPERTIES OF PRIME IDEAL GRAPHS OF COMMUTATIVE RINGS

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ABSTRACT

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The prime ideal graph of P in a finite commutative ring R with unity, denoted by Γ_P, is a graph with elements of R as its vertices and two elements in R are adjacent if their product is in P. In this paper, we explore some interesting properties of Γ_P. We determined some properties of Γ_P such as radius, diameter, degree of vertex, girth, clique number, chromatic number, independence number, and domination number. In addition to these properties, we study dimensions of prime ideal graphs, including metric dimension, local metric dimension, and partition dimension; furthermore, we examined topological indices such as atom bond connectivity index, Balaban index, Szeged index, and edge-Szeged index.

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1. INTRODUCTION

The concept of graphs has been used since the 17th century, namely in the Konigsberg bridge problem [1]. Although graph theory is a subject that has been appeared since a long time ago, graph theory is still an interesting topic to discuss today, because graphs can be used to illustrate discrete objects and the relationships between one object with another. As a result of this usage, graphs can be applied in various fields of mathematics and science.

In its development, graphs are used as a representation of mathematical systems namely groups, rings, and modules. Some graphs that can be used to represent a group are coprime graph [2], non-coprime graph [3], power graph [4], and intersection graph [5]. Graphs that can be used to represent a ring are zero divisor graph [6], prime graph [7], and Jacobson graph [8]. In addition, graphs that can be used to represent a module are annihilator graphs [9].

In 2022, Salih and Jund developed a graph used to represent a commutative ring with connection to a prime ideal. This graph was later named as prime ideal graph. They defined a prime ideal graph P of ring R, denoted by Γ_P, as a graph where the set of vertices is $R \setminus \{0\}$ and two vertices r_1, r_2 are adjacent if and only if $r_1 r_2 \in P$ [10]. Since the definition of prime ideal graph in that research is made to find relationships between prime ideal graphs and zero-divisor graphs, 0 is not included as a vertex in prime ideal graphs. In this study, we focus more on discussing properties of prime ideal graphs without looking at its relationship with other graph over another rings. We include 0 as a vertex in our definition of prime ideal graph in this study to give a more complete view of the base commutative ring as a graph. By including 0, the set of vertices of prime ideal graphs is all elements of commutative ring R and this also gives the ability to consider and analyze the ideal $\{0\}$ of R. This trivial ideal is a prime ideal if and only if the base ring R is an integral domain.

Based on the description above, as a continuation and generalization of the research in [10], the authors are inclined in discussing some properties of the prime ideal graph of the commutative ring including vertex degree, radius, diameter, chromatic number, clique number, independence number, dominance number and girth. The authors also study the dimensions of this prime ideal graphs of commutative rings. Graphs dimensions examined in this research including metric dimension, local metric dimension, and partition dimension. For each type of dimension, we give the minimum resolving sets. Furthermore, we examine some degree-based and distance-based topological indices on prime ideal graph of commutative ring i.e., atom bond connectivity index (ABC), Balaban index (J), Szeged index (Sz), and edge-Szeged index (Sz_e).

2. RESEARCH METHODS

This research used a literature study method which aimed to study properties of prime ideal graphs from recent research papers and identifying properties that have not been discussed in previous studies. This research was conducted as follows: first, recent papers are collected and discussed to get an overview of this topic. Next, we analyzed properties of prime ideal graphs that have not been discussed in the earlier papers. The final step was proving each conjecture we made for relating to those properties.

3. RESULTS AND DISCUSSION

This section provides a brief description of concepts in graph theory. After mentioning definitions that we need, we give the exact values of radius, diameter, dimensions, topological indices, and other properties of prime ideal graphs of commutative rings.

3.1 Basic Concepts

In this study, we consider only simple graphs which are undirected, with no edges connecting a vertex to itself or multiple edges. If two vertices are adjacent, then we assume there is only one edge that connect these two. The undirected part comes from the commutativity of the ring. Below is a list of definitions that we will study in this paper.

Definition 1. [1] Let Γ be a graph, $V(\Gamma)$ and $E(\Gamma)$ will denote the set of vertices and edges of Γ, respectively. Graph Γ is called empty if the vertex set is empty. The other common structure of graph is called a complete graph denoted by K_n, which is if every two distinct vertices in $V(\Gamma)$ are adjacent to each other.

Definition 2. [1] The degree of a vertex v, denoted by $\deg(v)$, is the number of edges which are incident to v; or in other words, it is the number of vertices adjacent to v.

Definition 3. [1] A subset Ω of $V(\Gamma)$ is called a clique if the induced subgraph of Ω is complete. The order of the largest clique in Γ is called its clique number, which is denoted by $\omega(\Gamma)$.

Definition 4. [1] If $u, v \in V(\Gamma)$, then $d(u, v)$ denotes the length of the shortest path between u and v. The largest distance between all pair of vertices of Γ is called the diameter of Γ and denoted by $diam(\Gamma)$.

Definition 5. [1] A set S of vertices of Γ is called a dominating set of Γ if every vertex in $V(\Gamma) \setminus S$ is adjacent to some vertex in S; the cardinality of a minimum dominating set is called a domination number of Γ and is denoted by $\gamma(\Gamma)$.

Definition 6. [1] The chromatic number of a graph Γ, denoted by $\chi(\Gamma)$, is the minimal number of colors needed to color the vertices in such a way that if two vertices are adjacent then they will have a different color.

Definition 7. [11] Let $W = \{w_1, w_2, \dots, w_k\}$ be an ordered subset of vertices in graph Γ. For $v \in V(\Gamma)$, the k-tuple $r(v|W) = \big(d(v, w_1), d(v, w_2), \dots, d(v, w_k)\big)$ of the distance between v and every vertex on W is called a representation of v with respect to W. The set W is called a resolving set of Γ if every two distinct vertices $x, y \in V(\Gamma)$ have a different representation with respect to W. A basis of Γ is a resolving set of Γ with the minimum number of vertices, and the metric dimension of Γ refers to its cardinality and is denoted by $\beta(\Gamma)$.

Definition 8. [12] If $r(x|W) \neq r(y|W)$ for every pair x, y of adjacent vertices in Γ, then W is called a local metric set of Γ. The minimum k for which Γ has a local metric k-set is the local metric dimension of Γ, denoted by $lmd(\Gamma)$. A local metric set of cardinalities $lmd(\Gamma)$ in Γ is a local metric basis of Γ.

Definition 9. [13] Suppose $\Delta = \{\Delta_1, \Delta_2, \dots, \Delta_l\}$ is an l-partition of $V(G)$. A partition representation of a vertex α with respect to Δ is the $l-$vector $(d(\alpha, \Delta_1), d(\alpha, \Delta_2), \dots, d(\alpha, \Delta_l))$, denoted by $r(\alpha|\Delta)$. Any partition Δ is referred as resolving partition if only if for every $\alpha_i \neq \alpha_j \in V(G)$, $r(\alpha_i|\Delta) \neq r(\alpha_j|\Delta)$. The smallest integer l is referred as the partition dimension $pd(G)$ of G if the l-partition Δ is a resolving partition.

Definition 10. [14] Suppose uv is an edge of Γ_P then atom bond connectivity index of Γ_P is defined as $ABC(\Gamma_P) = \sum_{uv \in E(\Gamma_P)} \sqrt{\frac{\deg(u) + \deg(v) - 2}{\deg(u)\deg(v)}}$.

Definition 11. [15] Let $D_{\Gamma_P}(u) = \sum_{v \in V(\Gamma_P)} d(u, v)$ and $q = |E(\Gamma_P)| - |V(\Gamma_P)| - 1$. Then Balaban Index ($J$ index) of Γ_P defined by $J(\Gamma_P) = \frac{|E(\Gamma_P)|}{q+1} \sum_{uv \in E(\Gamma_P)} \frac{1}{\sqrt{D_{\Gamma_P}(u) D_{\Gamma_P}(v)}}$.

Definition 12. [16] Suppose an edge $e = uv \in E(\Gamma_P)$. The values $n_1(e)$ and $n_2(e)$ are defined as the number of vertices which are closer to u and v respectively. Similarly, the quantities $m_1(e)$ and $m_2(e)$ are defined as the number of edges that are closer to u and v respectively. For an arbitrary edge $e = uv$ and vertex x, the distance between e and x is defined as $d(e, x) = \min\{d(u, x), d(v, x)\}$. Szeged index and edge-Szeged index are defined as $Sz(\Gamma_P) = \sum_{e \in E(\Gamma_P)} n_1(e) n_2(e)$ and $Sz_e(\Gamma_P) = \sum_{e \in E(\Gamma_P)} m_1(e)\, m_2(e)$.

Below is the definition of prime ideal graph that we used in this paper.

Definition 13. Let $(R, +, \circ)$ be a finite commutative ring and P be a prime ideal of R. The prime ideal graph of P in R, denoted by Γ_P, is a graph where the set of vertices is R and two vertices r_1, r_2 are adjacent if and only if $r_1 r_2 \in P$.

3.2 Properties of Prime Ideal Graphs

Let $(R, +, \circ)$ be a finite commutative ring with cardinality $|R| = \eta$ and P be a prime ideal of R with cardinality $|P| = \mu$. In this section, Γ_P always denotes prime ideal graph of P in commutative ring R. Based on Lagrange's theorem, since P can be seen as a subgroup of R, we have $\mu|\eta$. Therefore if $P \neq R$, then it must be $\mu \leq \eta/2$.

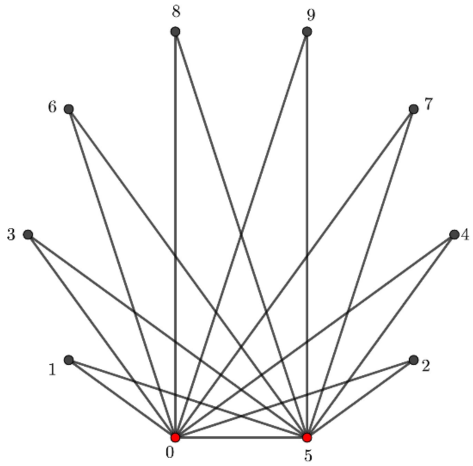
From the definition of prime ideal graph of a commutative ring, we can see that the number of vertices in this graph is $\eta = |R|$. Let $v \in P$ be a vertex in prime ideal graph Γ_P. For all $w \in \Gamma_P$, we have $v \circ w \in P$ since P is an ideal. Therefore, $v \in P$ is adjacent to all other vertices in Γ_P. Otherwise, if $v \notin P$ then v adjacent to all vertices in P and not adjacent to all $w \notin P$. Using this observation and assuming $|P| = \mu$ we get

$$\sum_{v \in \Gamma_P} \deg(v) = \mu(\eta - 1) + \eta\mu = \mu(2\eta - 1).$$

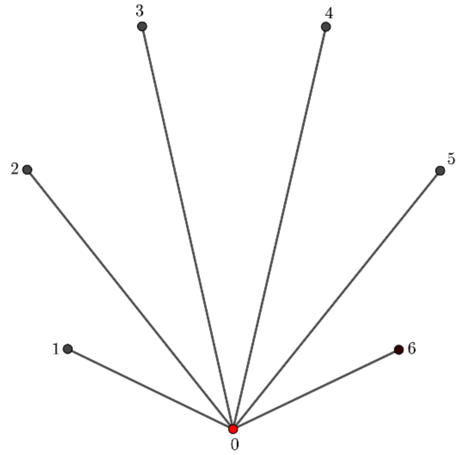
Therefore, the number of edges in Γ_P is given by $\mu(2\eta - \mu - 1)/2$.

Note that for a finite commutative ring R,the shape of prime ideal graph with R as the set of vertices depends on prime ideal P of R taken. If $P = R$ then $x \circ y \in P$ for all $x, y \in R$, hence we have $\Gamma_P \cong K_\eta$, a complete graph with $\eta = |R|$ vertices. If $P = \{0\}$ and R is an integral domain then $x, y \in R \setminus \{0\}$ implying $x \circ y \neq 0$ and so x and y is not adjacent in Γ_P. Therefore, for $P = \{0\}$, the resulting prime ideal graph is a star graph $K_{1,\eta-1}$ with η vertices. This can be seen in the example below.

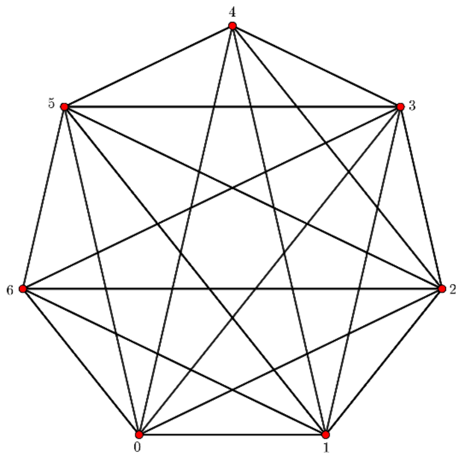
Example 1**.** Some examples of a prime ideal graph are given below. In the first figure, we illustrate the prime ideal graph of $P = \{0,5\}$ in the commutative ring $\mathbb{Z}_{10}$. The second figure is the prime ideal of $P = \{0\}$ in $R = \mathbb{Z}_7$. In the last figure, we take $P = R = \mathbb{Z}_7$.



(a)



(b)



(c)

Figure 1. **Prime ideal graph of (a) $P = \{0, 5\}$, $R = \mathbb{Z}_{10}$, (b) $P = \{0\}, R = \mathbb{Z}_7$, (c) $P = R = \mathbb{Z}_7$.**

On the following theorems, we analyze properties of prime ideal graph. Despite some of the properties investigated here, i.e., theorems that preceded metric dimension theorem, being the same with the one discussed in [10], there is a subtle difference on the graph that we define here. In this paper we consider the prime ideal graph that include zero element as one of its vertices, therefore that graph we discussed here is not identical to the graph considered in [10].

First, we described degree of vertices in a prime ideal graph of a commutative ring. The degree of a vertex v can be classified into two numbers depending on whether it is an element of P or not.

Theorem 1. *Let v be a vertex on Γ_P. Then*

$$\deg(v) = \begin{cases} \eta - 1, & v \in P \\ \mu, & v \notin P \end{cases}$$

Therefore, the largest degree of vertices on Γ_P is $\Delta = \eta - 1$ and the smallest degree is $\delta = \mu$.

Proof. Let $v \in P$. Based on the definition of ideal in a commutative ring, we have $v \circ w \in P$ for all $w \in R$. Therefore, v is adjacent to all other vertices in Γ_P. Thus, in this case $\deg(v) = \eta - 1$. Now let $v \notin P$. If $w_1 \in P$, then it is clear that $v \circ w_1 \in P$. Note that if $w_2 \notin P$, then $v \circ w_2 \notin P$. Therefore, v is adjacent to all $w_1 \in P$ and not adjacent to all $w_2 \notin P$. Thus, in this case $\deg(v) = |P| = \mu$.■

Now, by looking at distance of every two vertices, we can study the radius and diameter of a prime ideal graph of commutative ring. In Theorem 2 and Theorem 3, we give the exact values of radius and diameter.

Theorem 2. *Radius of Γ_P is 1. The Center set of Γ_P is P.*

Proof. Note that all vertices on P are adjacent to all vertices on Γ_P. Therefore, $\max_w d(v,w) = 1$ if $v \in P$ and $rad(\Gamma_P) = \min_v \max_w d(v,w) = 1$. Next, note that for $v, w \notin P$ we have $d(v,w) = 2$ so that $\max_w d(v,w) = 2$. Thus, center of Γ_P is given by $\left\{v \in \Gamma_P \middle| \max_w d(v,w) = 1\right\} = P$. So, we have proved that radius of Γ_P is 1 and its center is P.■

Theorem 3. *Diameter of Γ_P is given by*

$$diam(\Gamma_P) = \begin{cases} 1, & P = R \text{ or } R \cong \mathbb{Z}_2, \\ 2, & otherwise. \end{cases}$$

Proof. If $P = R$ then Γ_P is a complete graph K_η. Therefore $diam(\Gamma_P) = 1$. If $R \cong \mathbb{Z}_2$ then Γ_p form a path P_2 so that we get $diam(\Gamma_P) = 1$.

Next, let $P \neq R$ with $|R| \geq 3$ and $v, w \in \Gamma_P$. If $v \in P$ or $w \in P$ then v and w are adjacent to each other and thus $d(v,w) = 1$. If $v, w \notin P$ then v and w are not adjacent. Take an arbitrary $p \in P$, then p is adjacent to both v and w by the property of prime ideal. Therefore $d(v,w) = 2$. Since the distance of every two elements in Γ_P is at most 2, then $diam(\Gamma_P) = 2$.■

On the following theorem, we will consider the existence of a cycle subgraph in a prime ideal graph of a commutative ring. We can see that a prime ideal graph of a commutative ring is an acyclic graph if and only if $R \cong \mathbb{Z}_2$ or $P = \{0\}$.

Theorem 4. *Girth of a Γ_P is given by*

$$girth(\Gamma_P) = \begin{cases} \infty, & P = \{0\} \text{ or } R \cong \mathbb{Z}_2 \\ 3, & otherwise \end{cases}$$

Proof. If $R \cong \mathbb{Z}_2$ then there are two possible prime ideals, that is $P = R$ and $P = \{0\}$. Both cases give us path graph P_2 with no cycle. This implies that $girth(\Gamma_P) = \infty$ for $R \cong \mathbb{Z}_2$. We know that $P = \{0\}$ is a prime ideal if and only if R is an integral domain and in this case, the prime ideal graph is given by $\Gamma_P \cong K_{1,\eta-1}$. Thus, there is no cycle in the graph and the girth is ∞.

Let $\eta \geq 3$. If $P = R$ then Γ_P is a complete graph so that it contains a cycle of order 3. If $P \neq R$ then for $r \in R \backslash P$ and $p_1, p_2 \in P$, we have all these three vertices adjacent to each other. Therefore Γ_P always contain C_3 and its girth is equal to 3.■

Theorem 5. *The clique number Γ_P is equal to*

$$\omega(\Gamma_P) = \begin{cases} \eta, & P = R \\ \mu + 1, & otherwise \end{cases}$$

Proof. Note that every vertex in P adjacent to each other, hence vertices of P induce a complete graph K_μ. Suppose that $P \neq R$ and $r \in R \setminus P$. Since r is adjacent to all vertices in P then $\{r\} \cup P$ induce complete graph

$K_{\mu+1}$. If r' is another vertex in $R \setminus P$ then r' will not be adjacent to r. Therefore, $\{r, r'\} \cup P$ is not a complete graph. So, the clique number of Γ_P is $\mu + 1$ if $P \neq R$. If $P = R$ then Γ_P induce complete graph of order η.■

Using the above property, we can determine specifically the chromatic number of a prime ideal graph of commutative ring.

Theorem 6. *Chromatic number of Γ_P is given below.*

$$\chi(\Gamma_P) = \begin{cases} \eta, & P = R \\ \mu + 1, & otherwise \end{cases}$$

Proof. Since every vertex in P is adjacent to each other, this means that we need at least μ colors to make sure every vertex on P has different color. Thus, if $P = R$ then the chromatic number of Γ_P is equal to η.

If $P \neq R$ then since every vertex on $R\setminus P$ is not adjacent to each other, we can make every vertex in $R\setminus P$ have the same color. Therefore, we need at most $\mu + 1$ colors to do the coloring on Γ_P. Now, we show that $\mu + 1$ is the minimum number of colors. Suppose that we only need μ colors to do the coloring for Γ_P. According to Theorem 5, Γ_P contains a complete subgraph of order $\mu + 1$. This means there will be at least two vertices on this complete subgraph that have the same color. This is a contradiction with the assumption that we need only μ vertices. Therefore, the chromatic number of Γ_P if $P \neq R$ is equal to $\mu + 1$. ■

In the next two theorems, we find the largest number of vertices that is not adjacent to each other and the smallest number of vertices such that every vertex in Γ_P adjacent to one or more of these vertices.

Theorem 7. *The independence number of Γ_P is equal to* 1 *if $P = R$ and is equal $\eta - \mu$ if $P \neq R$.*

Proof. Note that if $P = R$, the graph Γ_P is a complete graph K_μ where every vertex is adjacent to another vertex. Therefore, in this case the independent number is equal to 1. Suppose $P \neq R$ and note that for two vertices $r_i, r_j \in R \setminus P$ we know that r_i and r_j is not adjacent. Using this observation, $R \setminus P$ is an independent set. We will show that $R \setminus P$ is a maximum independent set. Suppose W is an independent set with $|W| > |R \setminus P|$, then there is $p \in W$ with $p \in P$. Since $p \in P$ then p must be adjacent to all other vertex in W. This contradicts the assumption that W is an independent set. So, $R \setminus P$ is a maximum independent set and the independence number of Γ_P is $\eta - \mu$. ■

Theorem 8. *The domination number of Γ_P is equal to* 1.

Proof. Take an element $p \in P$. Note that every other vertex in Γ_P will be adjacent to p since $r \circ p \in P$ for all $r \in R$ from definition of ideal of a ring. So $\{p\}$ is a dominating set in graph Γ_P. Therefore, the domination number of Γ_P is equal to 1.■

In the following three theorems, given the metric dimension, local metric dimension, and partition dimension of prime ideal graph of commutative ring.

Theorem 9. *The metric dimension of Γ_P is $\eta - 1$ if $P = R$ and $\eta - 2$ if $P \neq R$.*

Proof. First, suppose that $P = R$, then Γ_P is a complete graph K_η. So, the metric dimension is given by $\eta - 1$ (see Theorem 3 in [11]). Suppose that $P \neq R$. Take $W \subset V(\Gamma_P)$ such that $|W| = \eta - 2$ and $V(\Gamma_P)\setminus W = \{a, b\}$ where $a \in P$ and $b \notin P$. Since a and b have different distance to all $r \in W \cap (R\setminus P)$ then they have distinct representation and hence W is a resolving set. Suppose that there is a resolving set W' with $|W'| = \eta - 3$. Then at least there are two elements in P (or in $R\setminus P$) but not in W'. These two elements will have the same representation with respect to W'. Hence W' is not a resolving set. This shows that in this case $\beta(\Gamma_P) = \eta - 2$.■

Theorem 10. *The local metric dimension of Γ_P is equal to $\mu - 1$ if $P = R$ and μ if $P \neq R$.*

Proof. First, suppose that $P = R$. In this case, Γ_P is a complete graph K_η. Therefore, the local metric dimension of Γ_P is equal to $\eta - 1$.

Suppose that $P \neq R$. In this case, take $W = P$. Let $v, w \in \Gamma_P$ are adjacent, then $v \in P$ or $w \in P$. If both $v, w \in P$ then they are in the set W and hence they will have distinct representation. If $v \in P$ and $w \notin P$ then representation of v contains a 0 and representation of w does not contain 0. So, W is a local metric set.

Note that all vertices in P have common closed neighborhood, so at least $\mu - 1$ vertices in P should be in local metric set. If W' is a minimum local resolving set with $|W'| = \mu - 1$ then without losing of generality, we can assume $W' = \{p_2, \dots, p_\mu\}$. But in this case p_1 and all $r \notin P$ have the same representation. So, the local metric dimension of Γ_P in this case is equal to μ.■

Theorem 11. *The partition dimension of Γ_P is equal to η if $P = R$ and $max\{\eta - \mu, \mu + 1\}$ if $P \neq R$.*

Proof. If $P = R$ then Γ_P is a complete graph K_η whose vertices are twin vertices. Therefore, every vertex must be in different partition. So, in this case $pd(\Gamma_P) = \eta$. If $P \neq R$, Γ_P contains a complete graph with $m + 1$ vertices and all vertices in $R \backslash P$ are twin vertices. Divide into two cases, when $\mu + 1 > \eta - \mu$ and when $\mu + 1 \leq \eta - \mu$.

Case I. If $\mu + 1 > \eta - \mu$ then the $\mu + 1$ vertices including μ vertices in P is the largest twin vertices equivalence class, hence $pd(\Gamma_P) \geq \mu + 1$. Define $(\mu + 1)$-partitions of $V(\Gamma_P)$ as $\Delta = \{L_1, \dots, L_{\mu+1}\}$ where $L_i = \{p_i, r_i\}$ for $i = 1, \dots, \mu - 1$, $L_\mu = \{p_\mu\}$, and $L_{\mu+1} = \{r_\mu\}$. Then p_i and r_i, $1 \leq i \leq \mu - 1$, will have different distance to r_μ (or p_μ) and thus Δ is minimum resolving partition of Γ_P. Therefore $pd(\Gamma_P) = \mu + 1$.

Case II. If $\mu + 1 \leq \eta - \mu$ then $\mu < \eta - \mu$. The largest twin vertices equivalence class is $R \backslash P$, hence $pd(\Gamma_P) \geq \eta - \mu$. Define $\Delta = \{L_1, \dots, L_{\eta-\mu}\}$ as a $(\eta - \mu)$-partitions of $V(\Gamma_P)$ where $L_i = \{r_i, p_i\}$ for $1 \leq i \leq \mu$ and $L_i = \{r_i\}$ for $i = \mu + 1, \dots, \eta - \mu$. For r_i, p_i where $1 \leq i \leq \mu$, their distance to $r_{\eta-\mu}$ is different and hence Δ is a minimum resolving partition. So, in this case $pd(\Gamma_P) = \eta - \mu$.■

The following theorems describing the atom bond connectivity index, Balaban index, Szeged index, and edge-Szeged index of any given prime ideal graph of commutative ring. First, we give the atom bond connectivity index of Γ_P. This topological index is used in chemical graph theory to study the stability of alkane as a structure.

Theorem 12. *The atom bond connectivity index of Γ_P is given by*

$$ABC(\Gamma_P) = \frac{\mu(\mu - 1)}{2(\eta - 1)}\sqrt{2\eta - 4} + \mu(\eta - \mu)\sqrt{\frac{\mu + \eta - 3}{(\eta - 1)\mu}}.$$

Proof. Suppose $uv \in E(\Gamma_P)$ is an edge in a prime ideal graph. If $u, v \in P$ then from Theorem 1, $d(u) = d(v) = \eta - 1$. Therefore, $\frac{1}{d(u)} + \frac{1}{d(v)} - \frac{2}{d(u)d(v)} = \frac{2}{\eta-1} - \frac{2}{(\eta-1)^2} = \frac{2\eta-4}{(\eta-1)^2}$. Note that there are $\frac{\mu(\mu-1)}{2}$ edges of this form. If $u \in P, v \notin P$ then from Theorem 1, $d(u) = \eta - 1$ and $d(v) = \mu$. Hence, we obtained $\frac{1}{d(u)} + \frac{1}{d(v)} - \frac{2}{d(u)d(v)} = \frac{1}{\eta-1} + \frac{1}{\mu} - \frac{2}{(\eta-1)\mu} = \frac{\mu+\eta-3}{\mu(\eta-1)}$. There are $\mu(\eta - \mu)$ edges of this form. By taking sum, the atom bond connectivity of prime ideal graph equal to

$$\frac{\mu(\mu - 1)}{2}\sqrt{\frac{2\eta - 4}{(\eta - 1)^2}} + \mu(\eta - \mu)\sqrt{\frac{\mu + \eta - 3}{\mu(\eta - 1)}}.$$

So, we have proved the desired result.■

Another index that can gives a description of chemical molecules as a graph is Balaban index. This topological index is a distance-based topological index. Balaban index gives us description of a graph based on distances from each vertex to another vertices in the graph.

Theorem 13. *The Balaban index Γ_P is given by*

$$J(\Gamma_P) = \frac{\mu(2\eta - \mu - 1)}{2\eta(\mu - 1) - \mu(\mu + 1)}\left(\frac{\mu(\mu - 1)}{2(\eta - 1)} + \frac{(\eta - \mu)\mu}{\sqrt{(\eta - 1)(2\eta - \mu - 2)}}\right).$$

Proof. Note that the number of edges in a prime ideal graph is $\frac{\mu}{2}(2\eta - \mu - 1)$. This gives $q = \frac{\mu}{2}(2\eta - \mu - 1) - \eta - 1$. Since all vertex $v \in P$ adjacent to all other vertices and all vertex $v \notin P$ adjacent only to vertices in P, then

$$D(u) = \begin{cases} \eta - 1, & u \in P, \\ \mu + 2(\eta - \mu - 1), & otherwise. \end{cases}$$

So that

$$J(\Gamma_P) = \frac{\frac{\mu}{2}(2\eta-\mu-1)}{\frac{\mu}{2}(2\eta-\mu-1)-\eta}\left(\frac{\mu(\mu-1)}{2}\frac{1}{\eta-1} + \mu(\eta-\mu)\frac{1}{\sqrt{\left(\mu+2(\eta-\mu-1)\right)(\eta-1)}}\right).$$

We have proved the statement in theorem above.■

The last topological index we study is Szeged index. This index is also introduced to measure the properties of drugs and chemical compounds.

Theorem 14. *The Szeged index of* Γ_P *is given by*

$$Sz(\Gamma_P) = \frac{\mu}{2}(\mu-1) + (\eta-\mu)^2.$$

And the edge-Szeged index of Γ_P *is given by*

$$Sz_e(\Gamma_P) = \frac{\mu(\mu-1)}{2}(\mu^2 - 2\mu\eta + 2\eta^2 - 4\eta + 4).$$

Proof. Let $e = uv$ is an arbitrary edge of Γ_P. If $u, v \in P$ then $n_1(e) = n_2(v) = 1$ and $m_1 = m_2 = \eta - 2$. If one of endpoint of e is not in P, say v, then the only vertex closer to v than u is v itself, vertices closer to u than v are all vertices in $R \backslash P \cup \{v\}$, and edges closer to u (or v) are all edges which incident to u (or v) but e. We obtain $n_1(e) = n_2(v) = 1$, $m_1 = \eta - 2$, and $m_2 = \mu - 1$. Hence, $Sz(\Gamma_P) = \frac{\mu}{2}(\mu-1) + (\eta-\mu)^2\mu$ and $Sz_e(\Gamma_P) = \frac{\mu}{2}(\mu-1)(\eta-2)^2 + \mu(\eta-\mu)(\eta-2)(\mu-1)$.■

4. CONCLUSIONS

In this paper, we have determined some values relating to properties of prime ideal graph of commutative ring including radius of graph, diameter, degree of vertex, girth, clique number, chromatic number, independence number, and domination number. We also determined metric dimension, local metric dimension, partition dimension, atom bond connectivity index, Balaban index, Szeged index, and edge-Szeged index of any given prime ideal graph.

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