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AN EXISTENCE AND UNIQUENESS OF THE SOLUTION OF SEMILINEAR MONOTONE ELLIPTIC EQUATION WITH THE DATA IN STUMMEL CLASSES

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ABSTRACT

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, f be a function in Stummel classes $\tilde{S}_\alpha(\Omega)$, where $\alpha = 1,2$, and

$$\begin{cases} -\mathrm{div}(A(x)\nabla \mathrm{u}) + g(u) = f\,, \\ \quad u \in W_0^{1,2}(\Omega), \end{cases}$$

be a semilinear monotone elliptic equation, where $A(x)$ is $n \times n$ symmetric matrix, elliptic, bounded, and $g\colon \mathbb{R} \to \mathbb{R}$ is non decreasing and Lipschitz. By proving a weighted estimation for a function in Stummel class with its weight in $W_0^{1,2}(\Omega)$, which allows us to use Stampacchia's lemma, we obtained the existence and uniqueness of the solution of this equation.

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1. INTRODUCTION

For $n \geq 3$, let Ω be a bounded open set of $\mathbb{R}^n$, with diameter l. For $a \in \mathbb{R}^n$ and $r > 0$, we define $\Omega(a,r) = \{y \in \Omega: |y-a| < r\}$.

Let $0 < \alpha < n$. The collection of all functions $f \in L^1(\Omega)$ for which

$$\eta_f(r) = \sup_{x\in\Omega} \int_{\Omega(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy < \infty, \forall r > 0,$$

is called **Stummel class** and denoted by $\tilde{S}_\alpha(\Omega)$. This classes have some inclusion properties with other function spaces and applications to the regularity of the solution of elliptic partial differential equations [1, 2, 3, 4, 5, 6, 7].

Let $0 \leq \lambda \leq n$. The collection of all functions $f \in L^1(\Omega)$ for which

$$\|f\|_{L^{1,\lambda}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{1}{r^\lambda} \int_{\Omega(x,r)} |f(y)| dy < \infty,$$

is called **Morrey space** and denoted by $L^{1,\lambda}(\Omega)$. The Morrey spaces have some inclusion relations with the Stummel classes and applications in study the theory of elliptic partial differential equations [2, 3, 8, 9, 10, 11, 12, 13, 14].

Recall that the Sobolev space $W^{1,2}(\Omega)$ is the collection of all functions $u \in L^2(\Omega)$ for which $|\nabla u| \in L^2(\Omega)$ and equipped by the norm $\|u\|_{W^{1,2}(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$. The closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$ is denoted by $W_0^{1,2}(\Omega)$. The space $W_0^{1,2}(\Omega)$ is a Hilbert space [15, p. 287].

In this paper, we are interest in investigating the existence and uniqueness of semilinear monotone elliptic equations, more precisely:

$$\begin{cases} -\text{div}(A(x)\nabla \text{u}) + g(u) = f, \\ \qquad u \in W_0^{1,2}(\Omega). \end{cases} \tag{1}$$

Here we assume that $A(x)$ is $n \times n$ symmetric matrix with properties: $A(x)$ is **elliptic**, that is, there exists $\nu > 0$ such that

$$A(x)\xi \cdot \xi \geq \nu|\xi|^2,$$

for all $\xi \in \mathbb{R}^n$; and $A(x)$ is **bounded**, that is, there exists $\kappa > 0$ such that

$$|M(x)| \leq \kappa,$$

for all $x \in \Omega$. In equation (1), $g: \mathbb{R} \longrightarrow \mathbb{R}$ is **non decreasing** and **Lipschitz**, that is, there exists $K_0 > 0$ such that

$$|g(s) - g(t)| \leq K_0|s-t|,$$

for all $s, t \in \mathbb{R}$. The data f in (1) is assumed belongs to $\tilde{S}_\alpha(\Omega)$, for $\alpha = 1$ or 2.

The existence, uniqueness, and the regularity of the solution and its gradient of the problem (1), for the linear case, that is, $g = 0$, and f belongs to the Morrey spaces $L^{1,\lambda}(\Omega)$, were studied by many authors, for example [10, 11, 6, 8, 14]. For $n-2 < \lambda < n$, the existence, uniqueness, and the regularity of the problem (1) obtained by [8, 6], but the detail proof for the existence and uniqueness of the problem was given by [10]. Meanwhile, for $0 < \lambda < n-2$, the problem (1) was studied by [11, 6, 14]. For the case $f \in \tilde{S}_2(\Omega)$ and the linear case $g = 0$, the existence, uniqueness, and the regularity of the problem (1) was studied by [8].

In this paper, we prove the existence and uniqueness of the problem (1) solution by assuming $f \in \tilde{S}_\alpha(\Omega)$ and $g \neq 0$. This generalized the result obtained by [6, 8, 10] since we work with semilinear equations, that is $g \neq 0$, and the inclusion of $\tilde{S}_\alpha(\Omega) \supset \tilde{S}_2(\Omega) \supset L^{1,\lambda}(\Omega)$, $n-2 < \lambda < n$ (see [2, 8]).

2. RESEARCH METHODS

We use the notations $\boldsymbol{C = C(n, m, \dots, p) > 0}$ to indicate that the constant $\boldsymbol{C}$ depends on $\boldsymbol{n, m, \dots, p}$. This positive constant value may be varied from line to line when its appear.

Our method uses the functional analysis tools, that is, the Stampacchia's lemma [16], combine with a weighted estimation for functions in Stummel class with its weight in $\boldsymbol{W_0^{1,2}(\Omega)}$, to handle a continuous linear mapping in the second variable and a bounded linear functional.

Let H be a Hilbert space with norm $\|\cdot\|$ and H^{-1} be the set of all liner functional on H. The map $\boldsymbol{B}: H \times H \to \mathbb{R}$ is called **continuous and linear in the second variable** if $u, \psi, \omega \in H$ and for every sequence $\{\psi_n\}_n$ in H such that $\lim_{n\to\infty} \|\psi_n - \psi\| = 0$, then $\lim_{n\to\infty} |\boldsymbol{B}(u, \psi_n) - \boldsymbol{B}(u, \psi)| = 0$ and $\boldsymbol{B}(u, \psi + \omega) = \boldsymbol{B}(u, \psi) + \boldsymbol{B}(u, \omega)$. Meanwhile, $F \in H^{-1}$ is called **bounded linear functional** if there exists a constant $C > 0$ such that $|F(u)| \leq C\|u\|$ for all $u \in H$.

We state the following Stampacchia's lemma [16] and refer to [17] for its proof.

Lemma 1 (Stampacchia). Let H be a Hilbert space and $\boldsymbol{B}: H \times H \to \mathbb{R}$ be a continuous and linear in the second variable, and there exists two constants $K_1 > 0$ and $K_2 > 0$ such that
(1) $|\boldsymbol{B}(u_1, \psi) - \boldsymbol{B}(u_2, \psi)| \leq K_1\|u_1 - u_2\|\|\psi\|, \ \forall u_1, u_2, \psi \in H,$
(2) $\boldsymbol{B}(u_1, u_1 - u_2) - \boldsymbol{B}(u_2, u_1 - u_2) \geq K_2\|u_1 - u_2\|^2, \ \forall u_1, u_2 \in H,$
then, for every bounded linear functional $F \in H^{-1}$, there exists a unique $u \in H$ such that $F(w) = \boldsymbol{B}(u, w)$ for every $w \in H$.

Now we define a map $\boldsymbol{B}: W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ by the formula

$$\boldsymbol{B}(u, \psi) = \int_\Omega A(x)\nabla u \cdot \nabla\psi + \int_\Omega g(u)\psi. \tag{2}$$

For $f \in \tilde{S}_\alpha(\Omega)$, we also define $F_f: W_0^{1,2}(\Omega) \to \mathbb{R}$ by the formula

$$F_f(\psi) = \int_\Omega f\psi. \tag{3}$$

Notice that, for every $u, \psi \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} |\boldsymbol{B}(u, \psi)| &\leq \int_\Omega |A(x)\nabla u \cdot \nabla\psi| + \int_\Omega |g(u)\psi| \\ &\leq C(\kappa, n)\int_\Omega |\nabla u||\nabla\psi| + \int_\Omega (C|u| + |g(0)|)|\psi| < \infty, \end{aligned}$$

since $A(x)$ is bounded and g is Lipschitz, where the second constant $C = C(K_0)$. This means (2) is well defined. Moreover, the function F_f defined by (3) is a linear functional on $W_0^{1,2}(\Omega)$ which is easily proved from the linearity of integration.

3. RESULTS AND DISCUSSION

The first result is that the map $\boldsymbol{B}$ defined by (2) is continuous and linear in the second variable.

Lemma 2. *The map* $\boldsymbol{B}$ *defined by (2) is continuous and linear in the second variable.*

Proof. Let $u, \psi \in W_0^{1,2}(\Omega)$ and $\{\psi_n\}_n$ be any sequence in $W_0^{1,2}(\Omega)$ such that $\lim_{n\to\infty} \|\psi_n - \psi\|_{W_0^{1,2}(\Omega)} = 0$. Notice that,

$$\left|\int_\Omega (A(x)\nabla u \cdot \nabla\psi_n - A(x)\nabla u \cdot \nabla\psi)\right| \leq \int_\Omega |A(x)\nabla u \cdot \nabla(\psi_n - \psi)|. \tag{4}$$

By using the boundedness of $A(x)$ and Hölder's inequality, the right-hand side of (4) is estimated as follows

$$\begin{aligned} \int_\Omega |A(x)\nabla u \cdot \nabla(\psi_n - \psi)| &\leq C(\kappa, n)\int_\Omega |\nabla u||\nabla(\psi_n - \psi)| \\ &\leq C(\kappa, n)\|\nabla u\|_{L^2(\Omega)}\|\nabla(\psi_n - \psi)\|_{L^2(\Omega)}. \end{aligned} \tag{5}$$

Joining together (4) and (5), we get

$$\left|\int_\Omega (A(x)\nabla u \cdot \nabla\psi_n - A(x)\nabla u \cdot \nabla\psi)\right| \leq C(\kappa, n)\|\nabla u\|_{L^2(\Omega)}\|\nabla(\psi_n - \psi)\|_{L^2(\Omega)}. \tag{6}$$

Furthermore, $|g(u)| \leq C(K_0)|u| + |g(0)| \in L^2(\Omega)$ since g is Lipschitz. Therefore

$$\begin{aligned}\left|\int_\Omega (g(u)\psi_n - g(u)\psi)\right| &\leq \int_\Omega |g(u)||\psi_n - \psi| \\ &\leq \int_\Omega (C(K_0)|u| + |g(0)|)|\psi_n - \psi| \\ &\leq \|C(K_0)|u| + |g(0)|\|_{L^2(\Omega)} \|\nabla(\psi_n - \psi)\|_{L^2(\Omega)}.\end{aligned} \tag{7}$$

We conclude from (6) and (7) that is

$$\begin{aligned}|\boldsymbol{B}(u,\psi_n) - \boldsymbol{B}(u,\psi)| &\leq \left|\int_\Omega (A(x)\nabla u\cdot\nabla\psi_n - A(x)\nabla u\cdot\nabla\psi)\right| + \left|\int_\Omega (g(u)\psi_n - g(u)\psi)\right| \\ &\leq C(\kappa,n)\|\nabla u\|_{L^2(\Omega)}\|\nabla(\psi_n - \psi)\|_{L^2(\Omega)} \\ &\quad + \|C(K_0)|u| + |g(0)|\|_{L^2(\Omega)}\|\nabla(\psi_n - \psi)\|_{L^2(\Omega)} \\ &\leq C\|\psi_n - \psi\|_{W^{1,2}(\Omega)},\end{aligned}$$

where $C = C(\kappa,n)\|\nabla u\|_{L^2(\Omega)} + \|C(K_0)|u| + |g(0)|\|_{L^2(\Omega)}$. This inequality gives us $\lim_{n\to\infty} |\boldsymbol{B}(u,\psi_n) - \boldsymbol{B}(u,\psi)| = 0$. This means $\boldsymbol{B}$ continuous in the second variable. The map $\boldsymbol{B}$ is linear in the second variable follows from the linearity property of ∇, the dot product $\cdot$, and the integration. □

Lemma 3. Let $\boldsymbol{B}$ be defined by (2). Then there exist two constants $K_1 > 0$ and $K_2 > 0$ such that
(1) $|\boldsymbol{B}(u_1,\psi) - \boldsymbol{B}(u_2,\psi)| \leq K_1\|u_1 - u_2\|_{W^{1,2}(\Omega)}\|\psi\|_{W^{1,2}(\Omega)}$, $\forall u_1, u_2, \psi \in W_0^{1,2}(\Omega)$,
(2) $\boldsymbol{B}(u_1, u_1 - u_2) - \boldsymbol{B}(u_2, u_1 - u_2) \geq K_2\|u_1 - u_2\|^2_{W^{1,2}(\Omega)}$, $\forall u_1, u_2 \in W_0^{1,2}(\Omega)$.

Proof. Let $u_1, u_2, \psi \in W_0^{1,2}(\Omega)$. The assumptions g Lipschitz, $A(x)$ bounded, and using Hölder's inequality, yields

$$\begin{aligned}|\boldsymbol{B}(u_1,\psi) - \boldsymbol{B}(u_2,\psi)| &= \left|\int_\Omega A(x)\nabla(u_1 - u_2)\cdot\nabla\psi + \int_\Omega \big(g(u_1) - g(u_2)\big)\psi\right| \\ &\leq C\int_\Omega |\nabla(u_1 - u_2)|\,|\nabla\psi| + C\int_\Omega |u_1 - u_2||\psi| \\ &\leq C\|\nabla(u_1 - u_2)\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} + C\|u_1 - u_2\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \\ &\leq C\|u_1 - u_2\|_{W^{1,2}(\Omega)}\|\psi\|_{W^{1,2}(\Omega)},\end{aligned} \tag{8}$$

where $C = C(\kappa, n, K_0) = K_1 > 0$. Furthermore, we observe that

$$\begin{aligned}&\boldsymbol{B}(u_1, u_1 - u_2) - \boldsymbol{B}(u_2, u_1 - u_2) \\ &= \int_\Omega A(x)\nabla u_1\cdot\nabla(u_1 - u_2) + \int_\Omega g(u_1)(u_1 - u_2) \\ &\quad - \int_\Omega A(x)\nabla u_2\cdot\nabla(u_1 - u_2)\int_\Omega g(u_2)(u_1 - u_2) \\ &= \int_\Omega A(x)\nabla(u_1 - u_2)\cdot\nabla(u_1 - u_2) + \int_\Omega \big(g(u_1) - g(u_2)\big)(u_1 - u_2) \\ &\geq \nu\int_\Omega |\nabla(u_1 - u_2)|^2 = \nu\|u_1 - u_2\|^2_{W^{1,2}(\Omega)},\end{aligned} \tag{9}$$

obtained by using $A(x)$ is elliptic and g is non-decreasing. Setting $K_2 = \nu$. From (8) and (9), the lemma is proved. □

The following lemma was stated in [5]. We will use this lemma to prove the next theorem.

Lemma 4. *If $f \in \tilde{S}_1(\Omega)$, then there exists a positive constant $C = C(n)$ such that,*

$$\int_{\Omega} |f\psi| \leq C\eta_f(l) \int_{\Omega} |\nabla\psi|,$$

for every $\psi \in W_0^{1,2}(\Omega)$.

Now we will prove a weighted estimation of a function in Stummel classes where the weight in compactly supported Sobolev spaces $W_0^{1,2}(\Omega)$.

Theorem 5. *Let* $\psi \in W_0^{1,2}(\Omega)$. *If* $f \in \tilde{S}_1(\Omega)$, *then there exists a positive constant* $C = C(n, l)$ *such that,*

$$\int_{\Omega} |f\psi| \leq C\eta_f(l)\|\psi\|_{W^{1,2}(\Omega)}.$$

If $f \in \tilde{S}_2(\Omega)$, *then there exists a positive constant* $C = C(n)$ *such that,*

$$\int_{\Omega} |f\psi| \leq C\left(\eta_f(l)\|f\|_{L^1(\Omega)}\right)^{\frac{1}{2}}\|\psi\|_{W^{1,2}(\Omega)}.$$

Proof. Let $f \in \tilde{S}_\alpha(\Omega)$. We first deal with the case $\alpha = 1$. By Lemma 4 and Hölder's inequality, there exists a positive constant $C = C(n, l)$ such that

$$\int_{\Omega} |f\psi| \leq C\eta_f(l) \int_{\Omega} |\nabla\psi| \leq C\eta_f(l)\|\psi\|_{W^{1,2}(\Omega)}.$$

Now, let $\alpha = 2$. Sub representation of ψ, Tonelli's theorem, and Hölder's inequality yield

$$\int_{\Omega} |f\psi| = \int_{\Omega} |f(x)\psi(x)|dx \leq C(n) \int_{\Omega} |f(x)| \left(\int_{\Omega} \frac{|\nabla\psi(y)|}{|x-y|^{n-1}} dy \right) dx$$

$$\leq C(n) \int_{\Omega} |\nabla\psi(y)| \left(\int_{\Omega} \frac{|f(x)|}{|x-y|^{n-1}} dx \right) dy \leq C(n)\|\nabla\psi\|_{L^2(\Omega)} \left(\int_{\Omega} \left(\int_{\Omega} \frac{|f(x)|}{|x-y|^{n-1}} dx \right)^2 dy \right)^{\frac{1}{2}}.$$

Notice that

$$\int_{\Omega} \left(\int_{\Omega} \frac{|f(x)|}{|x-y|^{n-1}} dx \right)^2 dy = \int_{\Omega} \left(\int_{\Omega} \frac{|f(z)|}{|z-y|^{n-1}} dz \right) \left(\int_{\Omega} \frac{|f(x)|}{|x-y|^{n-1}} dx \right) dy$$

$$= \int_{\Omega} \int_{\Omega} |f(z)||f(x)| \left(\int_{\Omega} \frac{1}{|z-y|^{n-1}|x-y|^{n-1}} dy \right) dxdz$$

$$\leq C(n) \int_{\Omega} \int_{\Omega} |f(z)||f(x)| \frac{1}{|x-z|^{n-2}} dxdz$$

$$= C(n) \int_{\Omega} |f(z)| \left(\int_{\Omega} \frac{|f(x)|}{|x-z|^{n-2}} dx \right) dz$$

$$\leq C(n)\eta_f(l)\|f\|_{L^1(\Omega)}.$$

Combining the last two inequalities, we have

$$\int_{\Omega} |f\psi| \leq C(n)\|\nabla\psi\|_{L^2(\Omega)} \left(\int_{\Omega} \left(\int_{\Omega} \frac{|f(x)|}{|x-y|^{n-1}} dx \right)^2 dy \right)^{\frac{1}{2}}$$

$$\leq C(n)\|\nabla\psi\|_{L^2(\Omega)}\left(C(n)\eta_f(l)\|f\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \leq C(n)\left(\eta_f(l)\|f\|_{L^1(\Omega)}\right)^{\frac{1}{2}}\|\psi\|_{W^{1,2}(\Omega)}.$$

The proof is complete. □

Because of Theorem 5, we obtain the following corollary which tells us that the linear functional defined by (3) is bounded.

Corollary 6. *The linear functional* $F_f: W_0^{1,2}(\Omega) \to \mathbb{R}$ *defined by (3) is bounded.*

Proof. Let $\psi \in W_0^{1,2}(\Omega)$. According to Theorem 5, we have

$$|F_f(\psi)| = \int_\Omega |f\psi| \leq C\|\psi\|_{W^{1,2}(\Omega)},$$

where $C = C\left(n, l, \eta_f(l)\right)$ or $C = C\left(n, \eta_f(l), \|f\|_{L^1(\Omega)}\right)$. □

With Lemma 3 and Corollary (6) in hand, we can now apply Stampacchia's lemma to obtain the existence and uniqueness of solution of the equation (1).

Theorem 7. There exists $u \in W_0^{1,2}(\Omega)$ which is unique solution of the equation (1) in the following sense:

$$\boldsymbol{B}(u,\psi) = \int_\Omega A(x)\nabla u \cdot \nabla\psi + \int_\Omega g(u)\psi = \int_\Omega f\psi = F_f(\psi),$$

for all $\psi \in W_0^{1,2}(\Omega)$.

4. CONCLUSIONS

The semilinear monotone elliptic equation (1) has a unique solution. This fact is proved by using a weighted embedding of a function in Stummel classes where the weight in compactly supported Sobolev spaces and Stampacchia's lemma.

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REFERENCES

[1] N. K. Tumalun, "Fungsi elemen kelas Stummel modulus terbatas tetapi bukan elemen dari kelas Stummel," *BAREKENG: J. Math. & App.*, vol. 15, no. 1, pp. 077-084, 2021.

[2] N. K. Tumalun, D. I. Hakim and H. Gunawan, "Inclusion between generalized Stummel classes and other function spaces," *Math. Inequal. Appl.*, vol. 23, no. 2, pp. 547-562, 2020.

[3] N. K. Tumalun and H. Gunawan, "Morrey spaces are embedded between weak Morrey spaces and Stummel classes," *J. Indones. Math. Soc.*, vol. 25, no. 3, pp. 203-209, 2019.

[4] N. K. Tumalun, H. Gunawan and J. Lindiarni, "On the inclusion between weak Lebesgue spaces and Stummel classes," in *5th ICRIEMS Proceedings*, Yogyakarta, 2018.

[5] N. K. Tumalun, D. I. Hakim and H. Gunawan, "Some function spaces and their applications to elliptic partial differential equations," *Matematički Vesnik,* vol. 75, no. 2, pp. 71-86, 2023.

[6] G. Di Fazio, "Regularity for elliptic equations under minimal assumptions," *J. Indones. Math. Soc.*, vol. 26, no. 1, pp. 101-127, 2020.

[7] M. D. Surnachev, "Estimates of solutions to the noncoercive Dirichlet problem for a second order elliptic equation in divergence form with drift from a Kato class," *Zap. Nauchn. Sem. POMI,* vol. 519, pp. 229-263, 2022.

[8] G. Di Fazio, "Poisson Equation and Morrey Spaces," *Journal of Mathematical Analysis and Applications*, vol. 163, pp. 157-167, 1992.

[9] S. Samko, "Morrey spaces are closely embedded between vanishing Stummel class," *Math. Ineq. Appl.*, vol. 17, no. 2, pp. 627-639, 2014.

[10] N. K. Tumalun, "An existence and uniqueness of the weak solution of the Dirichlet problem with the data in Morrey spaces," *BAREKENG: J. Math. & App.*, vol. 16, no. 3, pp. 829-834, 2022.

[11] N. K. Tumalun and P. E. A. Tuerah, "A regularity of the weak solution gradient of the Dirichlet problem for divergent form elliptic equations in Morrey spaces," *Aust. J. Math. Anal. Appl*, vol. 18, no. 2, pp. 14.1-14.7, 2021.

[12] P. E. A. Tuerah and N. K. Tumalun, "Some notes on the inclusion between Morrey spaces," *J. Math. Inequal.*, vol. 16, no. 1, pp. 355-362, 2022.

[13] N. K. Tumalun, "Proper inclusion between vanishing Morrey spaces and Morrey spaces," *Tensor: Pure and Applied Mathematics Journal*, vol. 2, no. 1, pp. 1-4, 2021.

[14] G. R. Cirmi and S. Leonardi, "Regularity results for the gradient of solutions of linear elliptic equations with L1,λ data," *Annali di Mat. Pura e Appl.*, vol. 185, no. 4, pp. 537-553, 2006.

[15] H. Brezis, Functional Analysis, Sobolev Spaces, and Partial Differential Equations, New York: Springer, 2011.

[16] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," *C. R. Acad. Sci. Paris*, vol. 258, pp. 4413-4416, 1964.

[17] L. Boccardo and G. Croce, Elliptic Partial Differential Equations: Exsistence and Regularity of Distributional Solutions, Berlin: De Gruyter, 2014.