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# GENERALIZATION OF VON-NEUMANN REGULAR RINGS TO VON-NEUMANN REGULAR MODULES

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ABSTRACT

*An element r in a commutative ring R is called regular if there exist s∈R such that rsr=r. Ring R is called vN (von-Neumann)-regular ring if every element is regular. Recall that for any ring R always can be considered as module over itself. Using the fact, it is natural to generalize the definition of vN-regular ring to vN-regular module. Depend on the ways in generalizing there will be some different version in defining the vN-regular module. The first who defined the concept of regular module is Fieldhouse. Secondly Ramamuthi and Rangaswamy defined the concept of strongly regular module of Fieldhouse by giving stronger requirement. Afterward Jayaram and Tekir defined the concept of vN-regular module by generalizing the regular element in ring to regular element in R-module M. In this paper we investigate the properties of each module regular and the linkages between each vN-regular module.*

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## 1. INTRODUCTION

Throughout this paper all rings are assumed to be commutative with nonzero identity. In 1936, von-Neumann [4] introduced the definition of a regular von-Neumann element which is then written as vN-regular element. The definition of the vN-regular element is motivated by the unit element, that is, if $a$ is a unit element, then there is $a^{-1}$ such that $aa^{-1} = 1$. It means that $aa^{-1}a = a$. From that unit element, a definition of vN-regular element is formed. Then element $e \in R$ is called idempotent element if fulfill $e^2 = e$. And for any set that generated by $a$ we always denote as $\langle a \rangle$. Now, let $R$ be a ring, $a \in R$ is called a vN-regular element if there exist $b \in R$ such that $a = aba$. Now, it can be seen that the definition of vN-regular elements is a generalization of the definition of unit elements. A ring is called vN-regular if for every element it is a vN-regular element. Furthermore, since each ring can be viewed as a module on its own, it is possible to generalize the definition of a vN-regular ring to a vN-regular module. The generalization of vN-regular ring to vN-regular module, there is more than one definition which will be discussed later in this paper.

In 1969, Fieldhouse [4] introduced the generalization of the vN-regular ring, namely as F-vN regular module. A module $M$ over ring $R$ is called a F-vN regular module if for every submodule $N$ in $M$, $rM \cap N = rN$ for any $r \in R$. Furthermore, Rawamurthi and Rangaswary [7] also defined the definition of vN-regular module which will called as RR-vN regular module. A module $M$ over R is called RR-vN regular module if for any submodule in M is strongly pure, that is $N$ is called strongly pure submodule in $M$ if for any $a \in N$ there is $R$-homomorphism $f: M \to N$ such that $f(a) = a$. Whereas, Jayaram and Tekir [5] in forming the vN-regular module definition firstly defining the definition of $M$-vN regular element. Let $M$ be a module of $R$, then $a \in R$ is called a $M$-vN regular if $aM = a^2M$. If for every $m \in M$ there exist $M$-vN regular element such that $Rm = aM$ then $M$ is called JT-vN regular module. And for the last, Anderson, et al [2] introduces a new definition of vN-regular modules by weakening the requirements of vN-regular modules defined by Jayaram. Furthermore, he is also describing the relationship between each of the four vN-regular modules that have been defined.

This paper will explain in more detail the proofs of the properties that had been proven before. According to Adam et al (2022), if $M$ is a finitely generated module over ring $R$ then module $M$ is a regular JT-vN module if and only if $M$ is a strong duo reduced multiplication module. The consequence of this property is that a ring is vN-regular ring if and only if it is a strongly reduced duo ring. We also using [11]–[14] for help us to prove the properties. Furthermore, we give a new property that is the linkages between JT-vN regular and total quotient of module.

## 2. RESULTS AND DISCUSSION

Firstly, we will introduce the definition and some properties of von-Neumann regular ring. Element $a \in R$ is called vN-regular element if there exists $b \in R$ such that $aba = a$. It is clear if $a$ is unit in ring $R$ then a is vN-regular element because there exist $b$ such that $ab = 1$ which means $aba = a$. Ring $R$ is called ring vN-regular if every element is a vN-regular element. Based on [8] also we have Propositions 1, 2, and 3.

**Proposition 1.** [8] *Let $p$ be idempotent element in ring $R$ then for any $a \in \langle p \rangle$, $a = pa$.*

**Proof.** It is clear that $pa \in \langle p \rangle$. Since $a \in \langle p \rangle$ there exist $r \in R$ such that $a = pr$. Then we have,

$$pa = ppr = p^2r = pr = a.$$

**Proposition 2.** [8] *An element $a$ in ring $R$ is vN-regular element if and only if there exists idempotent element $p$ such that $\langle a \rangle = \langle p \rangle$.*

**Proof**. Because of $\langle p \rangle = \langle a \rangle$ then $p \in \langle a \rangle$, it means there exist $x \in R$ such that $p = ax$. On the other hand, we also had the fact that $p$ is idempotent element, it means $p^2 = p$. So $p^2 = (ax)^2 = axax = p = ax$. According to Proposition 1 $a \in \langle p \rangle$ implies that $a = pa = axa$. Furthermore, assume for any $a \in R$ there exist $x \in R$ such that $axa = a$ with $p = ax$, then for any $s \in \langle a \rangle$, can be written as $s = ar_1$ for some $r_1 \in R$. And we have $s = ar_1 = (axa)r_1 = (pa)r_1 = ps$, so $s \in (p)$ it means $(a) \subseteq (p)$. With the similiar way it's easy to see $(p) \subseteq (a)$.

**Proposition 3.** [8] *Element $a$ in ring $R$ is vN-regular if and only if $\langle a \rangle = \langle a^2 \rangle$.*

**Proof.** Let $a \in R$ be a vN-regular element, then there exist $x \in R$ such that $axa = a$. For any $p \in \langle a \rangle \subseteq R$ there exist $r \in R$ such that $p = ar$. We get $p = ar = axar = a^2xr \in \langle a^2 \rangle$, so $\langle a \rangle \subseteq \langle a^2 \rangle$. And it's clear if $\langle a^2 \rangle \subseteq \langle a \rangle$. So, we can conclude that $\langle a^2 \rangle = \langle a \rangle$. To prove the otherwise assume that $\langle a^2 \rangle = \langle a \rangle$. It is clear that if $a \in \langle a \rangle$, so there exist $r \in R$ such that $a = a^2r = aar = ara$. It's proven that $a$ is vN-regular element.

### 2.1 Von-Neumann Regular Module

In this part we began to discuss about the definition and some basic properties of von-Neumann regular module.

**Definition 1. [7]** Submodule $\boldsymbol{N}$ of $\boldsymbol{M}$ over ring $\boldsymbol{R}$ is called pure submodule if $\boldsymbol{rM} \cap \boldsymbol{N} = \boldsymbol{rN}$ for any $\boldsymbol{r} \in \boldsymbol{R}$.

**Example 1**. Let $\mathbb{R}^2$ be an $\mathbb{R}$-module then $\boldsymbol{N} = \{(\boldsymbol{a}, \boldsymbol{0}) | \boldsymbol{a} \in \mathbb{R}\}$ is a pure submodule of module $\mathbb{R}^2$.

**Definition 2. [4]** Let $M$ be $R$-module. Then $M$ is called F (Fieldhouse)-vN regular module if every submodule in $M$ is pure submodule.

**Example 2.** *Ring $R$ is vN-regular if and only if $R$ is F-vN regular as $R$-module. For any $a \in R$tehere exist $b \in R$ such that $aba = a$. Let $N$ be submodule in $R$-modul $R$. If $x \in aR \cap N$ then $x = ar = n$ for some $r \in R$ and $n \in N$ . Based on that fact we have $x = ar = (aba)r = ab(ar) = abn \in aN$ and it's clear if $aN \subseteq aR \cap N$. It means that $aR \cap N = aN$. We proved that $R$ is F-vN regular as $R$-module. For the otherwise we assume that $R$ is F-vN regular as $R$-module. For any $a \in R$ it's clear that $aR$ is submodule in $R$. It means that $aR \cap aR = a^2R$. According to **Proposition 3** $a \in R$ is vN-regular in $R$. Finally, we conclude that $R$ is vN-regular ring.*

**Definition 3. [7]** Submodule $N$ of $M$ over ring $R$ is called strongly pure submodule if for any $n \in N$ there exist $R$-homomorphisme $f: M \to N$ such that $f(n) = n$.

**Definition 4. [7]** Let $M$ be $R$-module. Module $M$ is called RR (Ramamuthi and Rangaswamy)-vN regular module if every submodule in $M$ is strongly pure submodule.

According to **Proposition 3,** an element $a$ in ring $R$ vN-regular if and only if $aR = a^2R$. Using that fact, Jayaram and Tekir (2018) defined the generalize of element vN-regular as follows:

**Definition 5. [5]** Let $M$ be an $R$-module, then $a \in R$ is called M-vn regular if $aM = a^2M$.

Notice that if $R$ is a vN-regular ring then for every $a \in R$ there is always an element $b$ such that $Ra = aR = bR$. The existence of $b$ can be guaranteed because atleast there is $b = a$. On the other hand, $b$ must be a vN-regular element. Furthermore, we know that if $b$ is a vN-regular element, then $bR = b^2R$, so that $Ra = bR = b2R$. Based on this fact Jayaram and Tekir (2018) defined the generalize definition of the vn regular module as follows.

**Definition 6. [5]** Let $M$ be a module over ring $R$, then $M$ is called as JT (Jayaram and Tekir)-vN regular module if for every $m \in M$ there exist element $M$-vN regular $a \in R$ such that $Rm = aM$

We give some example of module JT-vN regular.

**Example 3.** Every simple $R$-module M is $JT - vN$ regular. Let $m \in M$, then $Rm\, Rm = \{0\} = 0M$ or $Rm = M = 1M$. On the other hand, we have $0\mathrm{M} = 0^2\mathrm{M}$ and $1\mathrm{M} = 1^2\mathrm{M}$, It means that 0 and 1 is M-vN regular. So M must be JT-vN regular module.

**Example 4.** Let $(G, +)$ be abelian group with prime number order $p$. It's clear that $G$ is module over ring $\mathbb{Z}$. On the other hand, we also know that $G$ is cyclic group with order $p$. Let $G = \langle g \rangle$ then there exist isomorphism $\mathrm{f}: \mathrm{G} \to \mathbb{Z}_\mathrm{p}$ with $f(g) = 1$. We know that $\mathbb{Z}_\mathrm{p}$ is a simple module over $\mathbb{Z}$. So $\mathbb{Z}_\mathrm{p}$is JT-vN regular $\mathbb{Z}$-module, and we conclude that G also JT-vN regular $\mathbb{Z}$-module.

**Example 5.** Let $R$ be a ring and $P$ is maximal ideal in $R$. It is clear that $M = R/P$ is R-module. And now we will show that M is JT-vN regular R-module. We know that if M is a field. Then all of the ideal in M only 0 and itself. It means that if we consider M as a module then submodule in M is only 0 and M itself. Based on that fact we proved that M is simple module which mean M is JT-vN regular R-module.

Now we introduce the weakening definition of JT-vN regular module that is Anderson et al. (2019) vN-regular module version. In this paper we will call Anderson et al. (2019) vN-regular module version with ASJ (Anderson, Sangmin, Juett)-vN regular module.

**Definition 7.** [2] Module $M$ over $R$ is called ASJ-vN regular if every element in $R$ is $M$-vN regular.

**Example 6.** Boolean ring is ASJ-vN regular module over itself.

Let $M$ be an $R$-module. According to [6], $M$ is called multiplication module if for any submodule $N$ in $M$ there exist ideal $I$ in $R$ such that $N = IM$. Also, $Z_R(M) = \{r \in R | rm = 0 \text{ } for \text{ } some \text{ } 0 \neq m \in M\}$ will be denotes as the set of zero divisor of $R$ on $M$. Furthermore, if $N, K$ is submodules of $M$ then the ideal $\{a \in R | aK \subseteq N\}$ will denoted as $(N: K)$. It's clear that $(0: M) = \{a \in R | aM \subseteq 0\}$ is annihilator of $M$. Now, let $e \in R$ be idempotent element then we have $e = e^2$ which mean $e - e^2 = 0$. So, we have $e - e^2 \in (0: M)$. But for the otherwise it's not always true. For the example let $M = \mathbb{Z}_4$ be a module over $\mathbb{Z}$, then $4 - 4^2 = -12 \in (0: Z4)$ but $4 \neq 4^2 \in \mathbb{Z}$. Based on that explanation, a definition of the weak idempotent element can be constructed.

**Definition 8.** [5] Let $M$ be $R$-module, then $e \in R$ is called weak idempotent if $e - e^2 \in (0: M)$.

**Definition 9.** [5] $R$-module $M$ is called distributive colon if $(N: M) + (K: M) = (N + K: M)$ for any submodule $N, K$ in $M$

**Definition 10.** [5] Ideal $I$ and $J$ in $R$ is called complemented if $I \cap J = 0$ and $I + J = R$. Similarly, Submodule $N_1, N_2$ in $R$-module $M$ is called complemented if $N_1 \cap N_2 = 0$ and $N_1 + N_2 = M$.

We denote all ideals of ring $R$ as $L(R)$ and all submodules of $M$ $R$-module as $L_R(M)$. In this section we give some properties of each vN-regular module version. And also, we prove the main theorem that we will discuss in this paper.

### 2.2 The Properties of vN-regular Module

**Proposition 4.** [5] *Let $M$ be an $R$-module. If $e \in R$ is weak idempotent then $e$ is $M$-vN regular.*

**Proof.** $(e - e^2) \in (0: M) \Rightarrow (e - e^2)m = 0$ for all $m \in M$. It implies $em = e^2 m$ for all $m \in M$, then we have $eM = e^2 M$.

**Proposition 5.** [5] *Let $M$ be a multiplication $R$-module. If $J(R)$ is Jacobson radical of $R$ then the following statement are equivalent*

1. *$M$ is JT-vN regular and $Z_R(M) \subseteq J(R)$.*
2. *$M$ is trivial or $R$ is quasi local and $M$ is simple.*

**Proof.**

1. If $M$ trivial then it's clear. Now, assume that $M$ is not trivial and $M$ is JT-vN regular. Let $0 \neq m \in M$. Since $M$ is JT-vN regular then $Rm = aM = a^2 M$ for some $a \in R$. Note that $Rm = aM = a^2 M = a(aM) = a(Rm) = (Ra)m$. Hence $(R - Ra)m = 0$ which mean $(R - Ra) \subseteq Ann(m)$. Based on that fact we have $R \subseteq Ann(m) + Ra$ that implies $R = Ann(m) + Ra$. On the other hand, we know that $Ann(m) \subseteq Z_R(M) \subseteq J(R)$. Consequently $R = Ra$ that implies $Rm = M$ whch mean $M$ is simple. Since $ann(m)$ is maximal ideal and fulfill $Ann(m) \subseteq Z_R(M) \subseteq J(R)$ then $R$ must be quasi local.

2. To prove the other way is obvious by **Example 3.**

**Proposition 6.** [2] *Let $M$ be an $R$-module. If $R$ is vN-regular ring then $M$ is F-vN regular module.*

**Proof.** Let $N$ be a submodule in $M$ and $r \in R$. We will show that $rM \cap N = rN$. Let $x \in rM \cap N$ then $x = rm = n$ for some $m \in M$ and $n \in N$. Since $R$ is a vN-regular ring then there exist $s \in R$ such that $rsr = r$. Therefore $rsn = rs(rm) = (rsr)m = rm = n$. Clearly that $sn \in N$, which mean $r(sn) \in rN$. On the other hand, it's obvious if $rN \subseteq rM \cap N$. So, we proved that $rM \cap N = rN$.

The converse of **Proposition 6** is not always true, for the example:

**Example 7.** Module $\mathbb{Z}_6$ over $\mathbb{Z}$ is F-vN regular because of any submodule in M is pure but there is no $r \in \mathbb{Z}$ that fulfill $2.r.2 = 2 \in \mathbb{Z}$. This mean $\mathbb{Z}_6$ is F-vN regular module but $\mathbb{Z}$ is not vN-regular ring.

**Proposition 7.** [5] *Let $M_1, M_2$ be $R$-module and $f \in Hom_R(M_1, M_2)$. If $M_1$ is JT-vN regular module then $Im(f)$ is also JT-vN regular module.*

**Proof.** For any $m' \in Im(f)$ there exist $m \in M$ such that $m' = f(m)$. Because of $M_1$ is JT-vN regular module then there exist $M$-vN regular element $a \in R$ such that $Rm' = R(f(m)) = f(Rm) = f(aM) = f(a^2 M)$.

Based on that fact then we have $Rm' \ af(M) = a(Im(f)) = a^2(f(M)) = a^2(Im(f))$. Hence, we proved that properties.

**Corollary 1. [5]** *If $M$ is JT-vN regular $R$-module, and $N$ is submodule of $M$. Then module quotient $M/N$ is also JT-vN regular module.*

Furthermore, we give the necessary and sufficient condition an element is $M$-vN regular element.

**Lemma 1. [5]** *Let $M$ be finitely generated $R$-module. An element $a \in R$ is $M$-vN regular if and only if $\langle a\rangle + (0{:}\,M) = \langle a^2\rangle + (0{:}\,M)$*

**Proof.** If $a \in R$ is $M$-vN regular, then $aM = a^2M$. It implies that $\langle a\rangle M = \langle a\rangle\langle a\rangle M$. Because of $M$ is finitely generated module then $\langle a\rangle M$ is also finitely generated module. According to **[3, Corollary 2.5]** then we have $(1 + ra)\langle a\rangle M = 0$ for some $r \in R$. It means $(a + ra^2)(m) = 0$ for any $m \in M$. Based on that fact we have there exist $r_1 \in (0{:}\,M)$ such that $a + ra^2 = r_1 \Rightarrow a = -ra^2 + r_1$ and implies $\langle a\rangle + (0{:}\,M) \subseteq \langle a^2\rangle + (0{:}\,M)$. On the other hand, it's clear that $\langle a^2\rangle + (0{:}\,M) \subseteq \langle a\rangle + (0{:}\,M)$. Then finally we have $\langle a\rangle + (0{:}\,M) = \langle a^2\rangle + (0{:}\,M)$. For the otherwise we assume that $\langle a\rangle + (0{:}\,M) = \langle a\rangle + (0{:}\,M)$. Then there exist $r_1 \in R$ and $m \in (0{:}\,M)$ such that $a = a^2r_1 + m$. It means $aM = (a^2r_1 + m)M = a^2r_1M \subseteq a^2M$. On the other hand, it's clear that we have this implication $a^2M \subseteq aM$. So, we proved that $aM = a^2M$ or $a$ is $M$-vN regular.

**Lemma 2. [5]** *If $M$ is finitely generated $R$-module. Then $a \in R$ is $M$-vN regular if and only if $aM = eM$ for some weak idempotent element $e \in R$.*

**Proof.** Let $a \in R$ be a $M$-vN regular. Then $aM = a^2M$, based on Lemma we have $\langle a\rangle + (0{:}\,M) = \langle a^2\rangle + (0{:}\,M)$. It's clear that $a \in \langle a\rangle + (0{:}\,M)$, so there exist $r \in R$ and $r_1 \in (0{:}\,M)$ such that $a = a^2r + r_1$. Let $e = ar$ then we have $e - e^2 = ar - a^2r^2 = r_1r \in (0{:}\,M)$. It means $e$ is weak idempotent element. Because of $e = ar$ so it's clear that $\langle e\rangle \subseteq \langle a\rangle$ it implies $\langle e\rangle + (0{:}\,M) \subseteq \langle a\rangle + (0{:}\,M)$. On the other hand, we also have $a = a^2r + r_1 = a(ar) + r_1 = ae + r_1 = ea + r_1$. It implies that $\langle a\rangle + (0{:}\,M) \subseteq \langle e\rangle + (0{:}\,M)$. So, we have $\langle a\rangle + (0{:}\,M) = \langle e\rangle + (0{:}\,M) \Rightarrow \langle a\rangle M + (0{:}\,M)M = \langle a\rangle M = \langle e\rangle M + (0{:}\,M)M = \langle e\rangle M$. And it implies $eM = aM$. For the otherwise we assume that $aM = eM$ for some weak idempotent $e \in R$. Because of $e \in R$ is weak idempotent then $e - e^2 \in (0{:}\,M) \Leftrightarrow (e - e^2)M = 0 \Leftrightarrow eM = e^2M$. So, $aM = eM = e^2M = e(aM) = a(eM) = a(aM) = a^2M$. It means that $a$ is $M$-vN regular element.

**Lemma 3. [5]** *If $M$ is finitely generated $R$-module then these statements are equivalent:*

1. *$R/(0{:}\,M)$ is vN-regular ring.*
2. *Every element in $R$ is $M$-vN regular.*
3. *$IM \cap JM = IJM$ for any $I, J \in L(R)$.*
4. *$IM = IIM = I^2M$, for any $I \in L(R)$.*
5. *Every submodule in $M$ is pure submodule.*

**Proof.**

1. Because of $R/(0{:}\,M)$ is vN-regular ring, for any $a \in R$ there exist $x \in R$ such that $\bar{a} = \overline{axa} = \overline{a^2x}$. It means that $\bar{a} - \overline{a^2x} = \bar{0} \in R/(0{:}\,M)$ then implies $a - a^2x \in (0{:}\,M)$. Furthermore, because of $a - a^2x \in (0{:}\,M)$ then we have $aM + (0{:}\,M)M = aM = a^2xM + (0{:}\,M)M = a^2xM \subseteq a^2M$. On the other hand, it's clear that if $a^2M \subseteq aM$ so we have $aM = a^2M$. And we already proved that $a \in R$ is $M$-vN regular. Now, we will prove statement 1 by statement 2. We assume that every element in $R$ is $M$-vN regular. Let $a \in R$, since $a$ is $M$-vN regular then we have $aM = a^2M$. Based on **Lemma 1** then we have $\langle a\rangle + (0{:}\,M) = \langle a^2\rangle + (0{:}\,M)$. Consequently, there exist $r \in R$ and $r_1 \in R$ such that $a = a^2r + r_1a \Leftrightarrow a - a^2r = r_1 \in (0{:}\,M)$. Since $a - a^2r = r_1 \in (0{:}\,M)$ then we have $\bar{a} = \overline{a^2r} \in R/(0{:}\,M)$. Hence, quotient ring $R/(0{:}\,M)$ is vN-regular ring.
2. We will prove statement 3 by statement 1. Let $I, J \in L(R)$. It's obvious if $IJM \subseteq IM$ and $IJM \subseteq JM$ hence $IJM \subseteq IM \cap JM$. Now we will prove it, if $IM \cap JM \subseteq IJM$. According to **[9, Corollary 1.7]** we have $IM \cap JM = \Big(\big(I + (0{:}\,M)\big) \cap \big(J + (0{:}\,M)\big)\Big)M$. Let $a \in \big(I + (0{:}\,M)\big) \cap \big(J + (0{:}\,M)\big)$ then there exists $a_1 \in I, b_1 \in J$ and $r_1, r_2 \in (0{:}\,M)$ such that $a = a_1 + r_1 = b_1 + r_2$. It means that $a^2 = (a_1 + r_1)(b_1 + r_2) = a_1b_1 + a_1r_2 + b_1r_1 + r_1r_2 \in IJ + (0{:}\,M)$. Based on assumption, $R/(0{:}\,M)$ is vN-regular ring. It implies that $\langle a\rangle + (0{:}\,M) = \langle a^2\rangle + (0{:}\,M)$. Consequently, we have $a \in \langle a\rangle +$

$(0\colon M) = \langle a^2 \rangle + (0\colon M) \subseteq IJ + (0\colon M)$. Furthermore, we also have $\big(I + (0\colon M)\big) \cap \big(J + (0\colon M)\big) \subseteq IJ + (0\colon M)$ thus $IM \cap JM \subseteq IJM$. So, we proved that $IM \cap JM = IJM$.

3. It is easy to see that $Im = Im \cap Im = IIM = I^2M$. So, we have been proven statement 4 by statement 3. To prove statement 2 by statement 4, Let $a \in R$ then from the assumption we have $\langle a \rangle M = \langle a \rangle \langle a \rangle M$, hence $aM = a^2M$. It means $a \in R$ is $M$-vN regular element. Next, we prove statement 5 by statement 3. Let $N$ be a submodule in $M$ and $a \in R$. Since $M$ is multiplication submodule then according to description of proof **[5, Lemma 3]** we have $N = (N\colon M)M$. Based on the assumption, we also have $aM \cap N = aM \cap (N\colon M)M = a(N\colon M)M = aN$. Furthermore, we got $aM \cap N = aN$. So, $N$ is pure submodule.
4. Lastly, we prove statement 2 by statement 5. Let $a \in R$, clearly $aM$ is a submodule in $M$. Based on assumption, $aM$ is pure submodule. It means that $aM = aM \cap aM = a(aM) = a^2M$. So, we prove that $a \in R$ is M-vN regular element. By description above we have proving $1 \Rightarrow 2, 2 \Rightarrow 1, 1 \Rightarrow 3, 3 \Rightarrow 4, 4 \Rightarrow 2, 3 \Rightarrow 5, 5 \Rightarrow 2$ and it means we already proven **Lemma 3.**

Now we ready to prove the main theorem about JT-vN regular modules.

**Theorem 1. [5]** *If $M$ is finitely generated $R$-module then these statements are equivalent:*

1. *$M$ is JT-vN regular module.*
2. *$M$ is multiplication module and satisfies of any conditions of **Lemma 3**.*
3. *$M$ is colon distributive, and $Rm$ has complement in $L_R(M)$ for any $m \in M$.*
4. *Every submodule in $M$ is JT-vN regular module.*

**Proof.**

1. Let $m \in M$. Since $M$ is JT-vN regular module, then there exists $M$-vN regular $a \in R$ such that $Rm = aM$. Consequently, $Rm = \langle a \rangle M$. According to **[9, Proposition 1.1]** and **[5. Lemma 6]** $M$ is multiplication module and quotient ring $R/(0\colon M)$ is vN-regular ring. So, we conclude that statement 2 are true. Now we will prove statement 1 by statement 2. According to **Lemma 3** quotient ring $R/(0\colon M)$ is vN-regular ring. According to the assumption, $M$ is multiplication module. Then by **[5, Lemma 7]** we finally conclude that $M$ is JT-vN regular module.
2. We will be proving statement 3 by statement 1. Based on **Lemma 3** and the description of the proofing section in **[5, Lemma 2 (iii)]** clearly that $M$ is colon distributive. Now we will prove that $Rm$ has complement in $L_R(M)$ for any $m \in M$. Let $m \in M$. Since $M$ is JT-vN regular module then there exist $M$-vN regular $a \in R$ such that $Rm = aM$. Based on **Lemma 2** there exists weak idempotent $e \in R$ that fulfill $Rm = aM = eM$. Furthermore, according to **[5, Lemma 1]** $eM = Rm$ has complement in $L_R(M)$**.**
3. To prove statement 1 by statement 3, let $m \in M$. Since $Rm$ has complement in $M$ then by **[5, Lemma 3]** there exist weak idempotent $e \in R$ such that $Rm = eM$. Furthermore, by **Lemma 2** there exist $M$-vN regular element $a \in R$ such that $Rm = eM = aM$. Finally, we proved that $M$ is JT-vN regular.
4. Next, we proved statement 4 by statement 1. Let $N$ be a submodule in $M$. According to **[5, Lemma 6]**, quotient ring $R/(0\colon M)$ is vn-regular ring. Hence by **Lemma 3** $N$ is pure submodule. Let $n \in N$, since $M$ is JT-vN regular module then $Rn = aM = a^2M$ for some $M$-vN regular element $a \in R$. Based on that fact we have $aM \cap N = Rn \cap N = Rn$ and since $N$ is pure then $Rn = Rn \cap N = aM \cap N = aN$. Therefore, $Rn = Rn \cap N = a^2M \cap N = a^2N$. It means that $Rn = aN = a^2N$ or equivalently we say that $N$ is JT-vN regular module.
5. Proofing statement 1 by statement 4 is obvious from the assumption. So, we had $1 \Rightarrow 2, 2 \Rightarrow 1, 1 \Rightarrow 3, 3 \Rightarrow 1, 1 \Rightarrow 4, 4 \Rightarrow 1$.

**Theorem 2. [2]** *Let $M$ be a finitely generated $R$-module. Then $M$ is JT-vN regular module if and only if $M$ is F-vN regular module.*

**Proof.** Since $M$ is finitely generated JT-vN regular module, then it's satisfying of **Lemma 3** and **Theorem 1.** Hence every submodule in $M$ is pure. So, $M$ is F-vN regular module. Again, with the same reason. Since $M$ is finitely generated F-vN regular module then it's satisfying of **Lemma 3** and **Theorem 1.** So, $M$ must be JT-vN regular moule.

**Proposition 7.** *Let $M$ be non-trivial torsion free $R$-module. If $M$ is JT-vN regular then $M$ is RR-vN regular.*

**Proof.** Let $N$ be a submodule in $M$ and $n$ is any element in $N$. Since $M$ is JT-vN regular then there exist $r \in R$ such that $Rn = rM = r^2M$. By assumption we have $(r^2 - r)M = 0 \Leftrightarrow r^2 = r$. Consequently $n = 1.n = rm_1 = r^2m_1$ for some $m_1 \in M$. Defined homomorphism $f$ from $M$ to $N$ by $f(m) = rm$. Therefore $f(n) = f(rm_1) = rf(m_1) = r^2m_1 = rm_1 = n$. So $N$ is strongly pure submodule which is $M$ is RR-vN regular module.

In general, we have this implication.

**Theorem 3.** [2] *Let $M$ be $R$-module. Then,*

*RR-vN regular module ⇒ F-vN regular module ⇒ ASJ-vN regular module*

**Proof.** (RR-vN regular ⇒ F-vN regular) Let $M$ be RR-vN regular $R$-module. It means that every submodule in $M$ is strongly pure. On the other hand, from [7] we know that every strongly pure submodule is pure submodule. Therefore $M$ is F-vN regular module. (F-vN regular ⇒ ASJ-modul regular) Let $a \in R$. Since $M$ is F-vN regular module then every submodule in $M$ is pure submodule. Clearly that $aM$ is submodule in $M$ which mean $aM$ is also pure submodule. Hence $aM \cap aM = a(aM) = a^2M$. On the other hand, we have $aM \cap aM = aM$. Consequently, $aM = a^2M$.

Now, according to [1] we have the linkages between JT-vN regular module and reduced module. An $R$-module $M$ is called reduced module if for any $M \in M$ and $a \in R$ with $a^2m = 0$ implies $am = 0$. Whereas $M$ is called strongly duo module if $Tr(N, M) = \{\sum Im(f) | f \in Hom_R(N, M)\} = N$ for any submodule $N$ in $M$.

**Lemma 4.** [5] *If $M$ is finitely generated JT-vN regular $R$-module then $M$ is reduced module.*

**Proof.** Let $m \in M$ and $a \in R$ with $a^2m = 0$. This mean $a^2 \in (0\colon Rm)$. Since $M$ is JT-vN regular module then there exist M-vN regular element $e \in R$ such that $Rm = eM$. Therefore $a^2 \in (0\colon Rm) = (0\colon eM)$. Since $a^2 \in (0\colon eM)$ then $a^2em' = (a^2e)m' = 0$ for any $m' \in M$. Hence $a^2e \in (0\colon M)$. Based on **Lemma 3** and **Theorem 1**. $R/(0\colon M)$ vN-regular ring. So, there exist $r \in R$ such that $a + (0\colon M) = a^2r + (0\colon M)$ which is $a - a^2r \in (0\colon M)$. Since on that fact then we have $ea = ea - a^2re + a^2re = e(a - a^2r) + r(a^2e) \in (0\colon M)$. Therefore $eaM = aeM = aRm = 0$. Consequently $a(1.m) = am = 0$. Finally, we conclude that $M$ is reduced module.

**Theorem 4.** [5] *If $M$ is finitely generated $R$-module then $M$ is JT-Vn regular if and only if $M$ is multiplication reduced strongly duo module.*

**Proof.** Since $M$ is finitely generated JT-vn regular then by **Lemma 4** $M$ is reduced module. Again, by **Theorem 1**. $M$ is multiplication module. Furthermore, we used [10, Lemma 2.12] to prove $M$ is strongly duo module. Let $m, m_1 \in M$ with $Ann(m) \subseteq Ann(m_1)$. Since $M$ is JT-vN regular then there exist $M$-vN regular $a \in R$ such that $Rm = aM$. Furthermore, based on **Lemma 2** there exists weak idempotent $e \in R$ such that $Rm = aM = eM = e^2M$. Consequently, $m = em^2 = e^2m_2$ for some $m_2 \in M$. It implies $em_2 - e^2m_2 = (1 - e)(em_2) = (1 - e)m = 0$ which mean $(1 - e) \in Ann(m) \subseteq Ann(m_1)$. Hence $m_1 = m_1 - 0 = m_1 - (1 - e)m_1 = em_1 \in eM = Rm$. We prove that $M$ is multiplication reduced strongly duo module. For the otherwise, let $a \in R$. If $x \in aM$ then there exist $m \in M$ such that $x = am$. Since $M$ is reduced module then for any $r \in R$ with $r(a^2m) = a^2(rm) = 0$ always fulfill $a(rm) = r(am) = 0$. This means $Ann(a^2m) \subseteq Ann(am)$. Since $M$ is strongly duo module then by [10, Lemma 2.12] $x = am \in R(a^2m) \subseteq a^2M$. On the other hand, clearly that $a^2M \subseteq aM$. Therefore $a \in R$ is M-vN regular. Since $M$ is multiplication module, then by **Lemma 3** and **Theorem 1**, $M$ is JT-vN regular module.

Let us to introduce some noktation. Let $M$ be an $R$-module. If $S_M = R \backslash Z_R(M)$ then $Q_R(M) = S_M^{-1}R$ denotes the total quotient ring of $R$ with respect to $M$ and $Q(M) = S_M^{-1}M$ denotes the total quotient module of $M$.

**Theorem 5.** *If $M$ is JT-vN regular $R$-module then $Q(M)$ is also JT-vN regular module over $Q_R(M)$.*

**Proof.** Let $x \in Q(M)$, it means there exist $a \in M$ and $b \in S_M$ such that $x = \frac{a}{b}$. Since $M$ is JT-vN regular $R$-module then $Ra = rM = r^2M$ for some $r \in R$. Also, we know that $a = 1.a \in Ra$. This means there exists $m_1, m_2 \in M$ such that $a = rm_1 = r^2m_2$. Let $y = \frac{r}{b} \in Q_R(M)$. We will show that if $Q_R(M)x = yQ(M) =$

$y^2Q(M)$. For any $u \in Q_R(M)x$ we have $u = \frac{t}{s}x = \frac{t}{s} \cdot \frac{a}{b} = \frac{ta}{bs}$ for some $t \in R$ and $s \in S_M$. Based on this fact, we have $u = \frac{ta}{bs} = \frac{r}{b} \cdot \frac{tm_1}{s} = y\frac{tm_1}{s} \in yQ(M)$ and $u = \frac{ta}{bs} = \frac{r^2}{b^2} \cdot \frac{tm_2b}{s} = y^2\frac{tm_2b}{s} \in y^2Q(M)$. So, $Q_R(M)x \subseteq yQ(M)$ and $Q_R(M)x \subseteq y^2Q(M)$. For the otherwise, let $v_1 \in yQ(M)$ and $v_2 \in y^2Q(M)$. This means $v_1 = y\frac{p1}{s1} = \frac{r}{b} \cdot \frac{p_1}{s_1} = \frac{rp_1}{bs_1}$ and $v_2 = y^2\frac{p_2}{s_2} = \frac{r^2}{b_2} \cdot \frac{p_2}{s_2} = \frac{r^2p_2}{b^2s_2}$ for some $p_1, p_2 \in M$ and $s_1, s_2 \in SM$. We know that $rp_1 \in rM$ and $r^2p_2 \in r^2M$ hence there exist $r_1, r_2 \in R$ such that $ar_1 = rp_1$ and $ar_2 = r^2p_2$. Therefore $v_1 = \frac{rp_1}{bs_1} = \frac{ar_1}{bs_1} = \frac{r_1}{s_1} \cdot \frac{a}{b} \in Q_R(M)x$ and $v_2 = \frac{r^2p_2}{b_2s_2} = \frac{ar_2}{b_2s_2} = \frac{r_2}{bs_2} \cdot \frac{a}{b} \in Q_R(M)x$. Moreover, we proved that $Q_R(M)x = yQ(M) = y^2Q(M)$ which mean $Q(M)$ is JT-vN regular $Q_R(M)$-module.

## 3. CONCLUSIONS

According to the description above, we know that the definition of von-Neumann regular module can be constructed from von-Neumann regular ring. Depending on the way the construction, it can have a different definition. It will be interesting in the future if there is a new definition of von-Neumann regular module, and it should be linkages between all of the definition of von-Neumann regular module.

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