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PRESIMPLIFIABLE AND WEAKLY PRESIMPLIFIABLE RINGS

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ABSTRACT

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Let R be a commutative ring with identity. Two elements a and b in R are called to be associates if $a|b$ and $b|a$, or equivalently, if $Ra = Rb$. The generalization of associate relation in R has given the idea for definitions of presimplifiable and weakly presimplifiable rings. First of all, it will be given definitions of very strong associate relation, strong regular associate relation, very strongly associate ring, and strongly regular associate ring. The presimplifiable ring is a commutative ring with the condition that every nonzero element is a unit element. While the weakly presimplifiable ring is a commutative ring with the condition that every nonzero element is regular element. Furthermore, it is shown that the relationship between very strongly associate ring with presimplifiable ring and the linkage between strongly regular associate ring and weakly presimplifiable ring. It is obtained that R is a presimplifiable ring if and only if R is a very strongly associate ring. Meanwhile, R is a weakly presimplifiable ring if and only if R is a strongly regular associate ring. Then, it is shown that the correlation between presimplifiable and weakly presimplifiable rings to its polynomial ring $R[X]$ and its the formal power series ring $R[[X]]$. If R is a weakly presimplifiable ring, then $R[X]$ and $R[[X]]$ are also weakly presimplifiable rings. However, if R is a presimplifiable ring, then $R[[X]]$ is also a presimplifiable ring but always not valid for $R[X]$.

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1. INTRODUCTION

In ring theory, there are particular elements that can be attributed to the reducibility and primacy of an element. The elements are regular and unit elements. It motivates the availability of an association between the elements in the ring. Let R be a commutative ring with identity. Suppose that a and b are two nonzero elements in the ring R. According to Malik, Moderson, and Sen (1997) [5], a and b were defined to be associates, if $a|b$ dan $b|a$, or equivalently $Ra = Rb$.

Since the associate relation in a ring can be generalized, this motivated Anderson and Chun [4] as well as Mooney and Wang [2] to define presimplifiable and weakly presimplifiable rings. Let R be a commutative ring. Ring R is said to be presimplifiable ring, if $a = ab$ for all $a, b \in R$, implies $a = 0$ or $b \in U(R)$ [2]. Furthermore, ring R is said to be weakly presimplifiable ring, if $a = ab$ for all $a, b \in R$, implies $a = 0$ or $b \in reg(R)$ [4].

In [2], Mooney and Wang gave the definition of very strongly associate ring that can be associated with presimplifiable ring. Anderson and Chun [4] also provided the definition of strongly regular ring that can be associated with weakly presimplifiable ring, as well as provided further the definition of associate relation.

Therefore, this study discusses the properties of presimplifiable and weakly presimplifiable rings, as well as Alsaraireh [1] provided the correlation between presimplifiable and weakly presimplifiable rings to its polynomial ring $R[X]$ and its formal power series ring $R[[X]]$. It is assumed that ring R is a commutative ring with identity.

2. RESEARCH METHODS

The steps to solve the problem in problem formulation are as follows:

a. To discuss the properties and characteristics of presimplifiable and weakly presimplifiable rings, needed concept of very strongly associated and strongly regular associated rings.
b. To discuss the correlation between presimplifiable and weakly presimplifiable rings, necessary concepts related to the ideal on the presimplifiable and weakly presimplifiable rings and the role of presimplifiable and weakly presimplifiable rings in certain rings.

So that the research methods used in this study are as follows:

- Learn the basic concepts of regular associate relation in commutative rings.
- Understand the basic concepts of strongly associated ring, very strongly associate ring, and strong regular associate ring in commutative rings and their properties and characteristics.
- Learn the basic concepts of presimplifiable and weakly presimplifiable rings in commutative rings and their properties and characteristics.
- Understand the correlation between the ring with a very strong associates ring with presimplifiable ring and the correlation between the ring with a strong regular associates ring with weakly presimplifiable ring and its properties.

3. RESULTS AND DISCUSSION

This chapter will present the research results that include the presimplifiable ring and the weakly presimplifiable ring and their properties.

3.1 Presimplifiable Ring

Before discussing the definition of presimplifiable ring, several concepts related to presimplifiable ring will be given.

Definition 1. [4] Let R be a commutative ring with identity and let $a, b \in R$.

1. The a and b elements are said to be **associates**, denoted $a \sim b$, if $a|b$ and $b|a$, or equivalently $Ra = Rb$.
2. The a and b elements are said to be **strong associates**, denoted $a \approx b$, if $a = ub$ for some $u \in U(R)$.
3. The a and b elements are said to be **very strong associates**, denoted $a \cong b$, if $a \sim b$ and further when $a \neq 0$, $a = rb$ $(r \in R)$, implies $r \in U(R)$.

Furthermore, a lemma is given about the relationship of associates with the principal ideal in a ring.

Lemma 1. *Let R be a commutative ring with identity and let $a, b \in R$. The following conditions are equivalent.*

1. $a \sim b$
2. $(a) = (b)$, *with* $(a) = \{ra \mid r \in R\}$ *i.e. the principal ideal constructed by the element a in ring R.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$, $a \sim b$. Suppose that $b = sa$ and $a = rb$ for some $r, s \in R$. Then, $(a) = ra = rsb = (b)$. Hence $a \sim b \Rightarrow (a) = (b)$.
(2) $\Rightarrow$ (1) Let $a, b \in R$. Suppose that $(a) = (b)$ with $(a) = \{ra \mid r \in R\}$. Then, $(a) = (b) \Leftrightarrow Ra = Rb$. Thus $a|b$ and $b|a$. So $a \sim b$. □

Example 1. Let $\mathbb{Z}$ be an integral domain and let $a, -a \in \mathbb{Z}$. Then, $a| -a$ and $-a \mid a$, gives $a \sim -a$. It also obtained that $a \approx -a$ and $a \cong -a$ with $-1 \in \mathbb{Z}$ is a unit.

Example 2. Let $\mathbb{Q}$ be an integral domain and let $a, -a \in \mathbb{Z}$. Then, $a| -a$ and $-a \mid a$, gives $a \sim -a$. It also obtained that $a \approx -a$ and $a \cong -a$ with $-1 \in \mathbb{Z}$ is a unit.

Furthermore, it is given the relationship between associates, strong associates, and very strong associates written in the following theorems.

Theorem 1. *The associates $\sim$ in the commutative ring R with identity is an equivalence relation.*

Theorem 2. *The strong associates $\approx$ in the commutative ring R with identity is an equivalence relation.*

Theorem 3. *Let R be a commutative ring with identity and $a, b \in R$. If a and b are said to be very strong associates, then a strong associates with b.*

Proof. Let $a, b \in R$, $a \cong b$. If $a = 0$, then $b = 0$. If a and b are nonzero elements in R, then $a = rb$ for some $r \in R$. Since $a \cong b$, then $r \in U(R)$. So $a \approx b$. □

Theorem 4. *Let R be a commutative ring with identity and $a, b \in R$. If a and b are said to be strong associates, then a associates with b.*
Proof. Let $a, b \in R$, $a \approx b$. Suppose that $a = ub$ for some $u \in U(R)$, then $b|a$. Since $u \in U(R)$, there exists $u^{-1} \in R$ then $b = u^{-1}a$. Thus $a|b$. So $a \sim b$. □

Definition 2. [2] Let R be a commutative ring with identity. Ring R is said to be **very strongly associate ring**, if for any $a, b \in R$, $a \sim b$ implies $a \cong b$.

Example 2. Let $\mathbb{Q}[X]$ be a ring. Suppose that $f(X), g(X) \in \mathbb{Q}[X]$, with $f(X) = 9X^2 - 18X + 9$ and $g(X) = 2X^2 - 4X + 2$. Then $\mathbb{Q}[X]$ is very strongly associate ring.

Definition 3. [2] Let R be a commutative ring. Ring R is said to be **presimplifiable ring**, if $a = ab$ for all $a, b \in R$, implies $a = 0$ or $b \in U(R)$.

Example 3. The integral domain $\mathbb{Z}$ is presimplifiable ring.

We next show the properties that equivalent to the presimplifiable ring [3].

Theorem 5. *For a commutative ring R with identity, the following conditions are equivalent.*

(1). For all $a, b \in R$, $a \sim b \Rightarrow a \cong b$.
(2). For all $a, b \in R$, $a \approx b \Rightarrow a \cong b$.
(3). For all $a, b \in R$, $a \cong a$.
(4). R is presimplifiable ring.
(5). $Z(R) \subseteq 1 - U(R) = \{1 - u | u \in U(R)\}$.
(6). $Z(R) \subseteq J(R)$.
(7). For $0 \neq r \in R$, $sRr = Rr \Rightarrow s \in U(R)$

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$. Suppose that $a \sim b$. Since $a \in R$ is a nonzero element in R with $a = rb$ ($r \in R$), then $r \in U(R)$. Thus, $a \approx b$. So $a \cong b$.
(2) $\Rightarrow$ (3) For $a \in R$, $a \approx a$. So $a \cong a$.
(3) $\Rightarrow$ (4) Let $a \in R$. Assume that $a = ba$, then $a \cong a$. Hence $a = 0$ or $b \in U(R)$. So R is presimplifiable ring.
(4) $\Rightarrow$ (5) Let $a \in Z(R)$. Suppose that $za = 0$ with $0 \neq z \in R$. Then, $z = z - za = z(1 - a)$, so $(1 - a) \in U(R)$.
(5) $\Rightarrow$ (6) Let $a \in Z(R)$. For $r \in R$, $-ra \in Z(R)$ and hence $(1 + ra) \in U(R)$. Thus $a \in J(R)$. So $Z(R) \subseteq J(R)$.
(6) $\Rightarrow$ (7) Suppose that $0 \neq r \in R$ and $sRr = Rr$ for some $r \in R$. Then $r = str$ for some $s, t \in R$. Thus $r(1 - st) = 0$, then $(1 - st) \in Z(R) \subseteq J(R)$. Then, $st = 1 - (1 - st) \in U(R)$. So $s \in U(R)$.
(7) $\Rightarrow$ (1) Suppose that $a \sim b$ and $0 \neq a \in R$. Then $r \in R$, $a = rb$ and hence $ta = rtb = rta (t \in R)$. Thus $Ra = rRb = rRa$. So $a \cong b$. □

Proposition 1. [2] *Ring R is presimplifiable if and only if R is a very strongly associate ring.*

Proof. Let R be a commutative ring and let $a, b \in R$.
($\Rightarrow$) Let R be presimplifiable ring. Suppose that $a \sim b$, then by Lemma 1, $(a) = (b)$. We must show that $a \cong b$. If $a = 0$, then $b = 0$. Since $\sim$ is symmetric, then $a = 0$. If a and b are nonzero elements in R, then $a = rb$ for some $r \in R$. Since $(a) = (b)$, then $b = sa$ for some $s \in R$. Thus $a = (rs)a$ for some $r, s \in R$. Since R is presimplifiable ring, then $rs \in U(R)$. Thus, $a \cong b$, so R is very strongly associate ring.
($\Leftarrow$) Let R be very strongly associate ring. Suppose that $a = ba$ for some $a, b \in R$. We must show that $a = 0$ or $b \in U(R)$. Since $(a) = (a)$, $a \sim a$. Then R is very strongly associate ring, so $a \cong a$. If a is nonzero element in ring R, then $a = ba$ for some $b \in R$. Thus $b \in U(R)$. So R is presimplifiable ring. □

Furthermore, it will be given theorems regarding the relationship of presimplifiable ring with polynomial ring $R[X]$and formal power series ring $R[[X]]$.

Theorem 6. [1] *Let R be a commutative ring with identity. Ring $R[X]$ is presimplifiable if and only if $Z(R) \subseteq Nil(R)$.*

Proof. ($\Rightarrow$) Let $0 \neq a \in Z(R)$ and $R[X]$ is presimplifiable ring. Then there exists $0 \neq b \in R$ such that $ab = 0$. Hence $b(1 + ax) = b$, for some $0 \neq x \in R$. Since $R[X]$ is presimplifiable ring, then $(1 + ax) \in U(R)$. Thus, $a \in Nil(R)$. So $Z(R) \subseteq Nil(R)$.
($\Leftarrow$) Let $f(X) = \sum_{i=0}^{n} a_i X^i \in R[X]$ and $g(X) = \sum_{i=0}^{n} b_i X^i \in R[X]$ with $f(X) = f(X)g(X)$ and $f(X) \neq 0$. So, $f(X) = f(X)g(X) \Leftrightarrow f(X)\big(1 - g(X)\big) = 0$ and there is $r \in R$ such that $r\big(1 - g(X)\big) = 0$. For $r \neq 0$ then $rb_0 = r$, while $rb_0 = 0$ for all $i = 1,2, \dots, n$. Hence, $b_0 \in U(R)$ and $b_1, \cdots, b_n \in Z(R) \subseteq Nil(R)$. So, $g(X) \in U(R[X])$. Therefore, $R[X]$ is a presimplifiable ring. □

Next, it will be given the example of polynomial ring $R[X]$ which is not a presimplifiable ring if it is known R is presimplifiable ring but 0_R is not primary ideal of R.

Example 4. Let R be a presimplifiable ring with $\mathrm{R} = \mathrm{Z}_2(+)\mathrm{Z}_4$ where 0_R is not a primary ideal of R. Suppose that $\mathrm{a} = (\bar{0}, \bar{1}) \in \mathrm{R}$ and $\mathrm{f} = (\bar{1}, \bar{0}) + (\bar{1}, \bar{1})\mathrm{X} \in \mathrm{R[X]}$, then $\mathrm{a} \sim \mathrm{af}$,but $\mathrm{a} \not\approx \mathrm{af}$. By Theorem 3, then $\mathrm{a} \ncong \mathrm{af}$. Thus, R[X] is not very strongly associate ring. By Proposition 1, such that R[X] is not a presimplifiable ring.

Theorem 7. [1] *Let R be a commutative ring. Formal power series $R[[X]]$ is presimplifiable ring if and only if R is presimplifiable ring.*

Proof. ($\Rightarrow$) Let $f(X), g(X) \in R\big[[X]\big]$, $f(X) = g(X)f(X)$ and $f(X) \neq 0$. If $f(X) = a_0 + a_1X + a_2X^2 + \cdots$, then n be least non negative integer such that $a_n \neq 0$. Hence, $f(X) = a_nX^n + a_{n+1}X^{n+1} + \cdots$ and $f(X) = g(X)f(X)$ with $g(X) = b_0 + b_1X + b_2X^2 + \cdots$. Then, $a_n = b_0a_n$ with $a_n \neq 0$. So, $\mathrm{b}_0 \in \mathrm{U(R)}$ since R is presimplifiable ring. Then, $g(X) = b_0 + b_1X + \cdots + b_nX^n$ is a unit in $R[[X]]$. Therefore, $R[[X]]$ is a presimplifiable ring.
($\Leftarrow$) Let $x = xy$ for $x, y \in R$. Since $R \subseteq R\big[[X]\big]$, then $x = xy$ is also in a ring $R\big[[X]\big]$. Thus $x = 0$ or y is a unit in R since $R \subseteq R\big[[X]\big]$. So, R is a presimplifiable ring. □

We next give the relationship of formal power series in indeterminates $X_1, \dots, X_n$ and presimplifiable ring.

Theorem 8. [4] *Let R be a commutative ring. Ring $R\big[[X_1, \dots, X_n]\big]$ is presimplifiable if and only of R is presimplifiable ring.*

Proof. ($\Rightarrow$) Let $x = xy$ for $x, y \in R$. Since $R \subseteq R[[X_1, \dots, X_n]]$, then $x = xy$ is also in a ring $R[[X_1, \dots, X_n]]$. Thus $x = 0$ or y is a unit in R since $R \subseteq R[[X_1, \dots, X_n]]$. So, R is a presimplifiable ring.
($\Leftarrow$) Let $f \in Z(R[[X_1, \dots, X_n]])$. Then the constant term $a \in f$ lies in $Z(R)$. So $1 - a \in U(R)$. Then the constant term of $1 - f$ is a unit in $R[[X_1, \dots, X_n]]$. Thus $1 - f$ is a unit. So, $R[[X_1, \dots, X_n]]$ is presimplifiable ring. □

3.2 Weakly Presimplifiable Ring

Before getting into the discussion of the weakly presimplifiable ring and its properties, we will first give some concepts related to the weakly presimplifiable ring.

Definition 4. [4] Let R be a commutative ring and let $a, b \in R$.

1. The a and b are said to be **strong regular associate**, denoted $a \approx_r b$, if there exist regular elements $r, s \in R$ with $a = rb$ and $b = sa$.
2. The a and b are said to be **very strong regular associate**, denoted $a \cong_r b$, if $a \sim b$ and one of the following properties holds.
   a) $a = b = 0$.
   b) $a = rb$, r is regular.

Example 5. Let $R = C([0,3])$, the ring of continuous functions on $[0,3]$. Define $a(t), b(t) \in R$ by $a(t) = b(t) = 1 - t \in [0,1]$, $a(t) = b(t) = 0 \in [1,2]$, and $a(t) = -b(t) = t - 2 \in [2,3]$. Then, $a(t) \approx_r b(t)$ since there exists regular element $c(t) \in R$ with $c(t) = 1 \in [0,1]$, $c(t) = 3 - 2t \in [1,2]$, and $c(t) = -1 \in [2,3]$, such that $c(t)a(t) = b(t)$ and $c(t)b(t) = a(t)$.

Furthermore, it is given the relationship between associates, strong regular associates, and very strong regular associates written in the following theorem.

Theorem 9. *Let R be a commutative ring with identity and let $a, b \in R$. If $a \approx_r b$, then $a \cong_r b$.*

Proof. Let $a, b \in R$, $a \approx_r b$. Since $a = rb$ and $b = sa$ for some regular elements $r, s \in R$, then $a \mid b$ and $b \mid a$. Hence $a \sim b$. If $a = 0$, then $b = 0$. If a and b are nonzero elements in R, then $a = rb$ for some $r \in R$. Since $r \in R$ is regular, then $a \cong_r b$. Thus, $a \approx_r b \Rightarrow a \cong_r b$. □

Here are the theorems about the property of strong regular associates and very strong regular associates.

Theorem 10. *The strong regular associates $\approx_r$ in the commutative ring R with identity is an equivalence relation.*

Proof. Let $a \in R$. Since $a = 1 \cdot a$ with $1 \in R$ regular, then $a \approx_r a$. Thus $\approx_r$ is reflexive. Next, let $a, b \in R$. Suppose that $a \approx_r b$. By Definition 4, then $b \approx_r a$. Thus $\approx_r$ is symmetric. After that, let $a, b, c \in R$. Suppose that $a \approx_r b$ and $b \approx_r c$. For $a \approx_r b$, there exist regular elements $p, q \in R$ such that $a = pb$ and $b = qa$. For $b \approx_r c$, there exist regular elements $r, s \in R$ such that $b = rc$ and $c = sb$. Therefore, $a = (pr)c$ and $c = (sq)a$ with $rp, qs \in R$ regular elements, such that $a \approx_r c$. Thus $\approx_r$ is transitive. □

Theorem 11. *The very strong regular associates $\cong_r$ in the commutative ring R with identity is an equivalence relation if and only if $a = ba$ $\forall a, b \in R$, then $a = 0$ or b is regular.*

Proof. ($\Leftarrow$) Let $a, b \in R$ with $a \cong_r b$. Suppose that $a = ba$ then $a = 0$ or b is regular. We must show that $\cong_r$ is an equivalence relation. Let $a \in R$. We know that $a \sim a$. Since $a = 1 \cdot a$ with 1 is identity in R, such that $1 \in R$ is regular. So $a \cong_r a$. Thus $\cong_r$ is reflexive.
Let $a, b \in R$ with $a \cong_r b$. Suppose that $a \neq 0$, $b = sa$ for some $s \in R$. Since $\sim$ is symmetric, such that $b \sim a$. Hence $a = tb$ for some $t \in R$ since $a \sim b$. Then $a \cong_r b$, such that $sts \in R$ is regular. Hence, $s \in R$ is regular. So, $b \cong_r a$. Thus $\cong_r$ is symmetric.

Let $a, b, c \in R$. Suppose that $a \cong_r b$ and $b \cong_r c$. For $a \cong_r b$, $a \sim b$ and if $a = pb$ for some $p \in R$, then $p \in R$ is regular. For $b \cong_r c$, $b \sim c$ and if $b = qc$ for some $q \in R$, then $q \in R$ is regular. Since $\sim$ is transitive, such that $a \sim c$. Thus $a \cong_r c$, since $a \sim c$ and $a = (pq)c$ for regular element $pq \in R$.
($\Rightarrow$) Let $a, b \in R$ with $a \cong_r b$. Assume that $\cong_r$ is an equivalence relation. We now show that $a, b \in R$ with $a = ba$, then $a = 0$ or $b \in R$ is regular. Since $\cong_r$ is reflexive, such that $a \cong_r a$ for some $a \in R$. If a is nonzero element in R, then $a = ra$ for some $r \in R$. Thus $r \in R$ is regular.

□

Definition 5. [4] Let R be a commutative ring. Ring R is said to be **strongly regular associate ring**, if whenever $a \sim b$ for $a, b \in R$, $a \approx_r b$.

Example 6. (Theorem 12, [4]) The ring $C([a, b])$ is strongly regular associate.

Example 7. [4] Let R be a commutative ring that is not strongly associate. Let $M = \bigoplus_{\mathcal{M} \in maks(R)} R/\mathcal{M}$. Then R(+)M is not strongly regular associate ring. Note that, $Z(R(+)M) = \big(R - U(R)\big)(+)M = R(+) - U(R(+)M)$, so $R(+)M$ is total quotient ring. Since R is not strongly associate ring, it means that there exist $a, b \in R$ with $a \sim b$, but $a \not\approx b$. Then $(a, 0) \sim (b, 0)$, but $(a, 0) \not\approx_r (b, 0)$ in $R(+)M$. If $(b, 0) = (r, m)(a, 0)$ for some regular element $(r, m) \in R(+)M$, then $r \in U(R)$. So $a \approx b$, a contradiction.

Definition 6. [4] Let R be a commutative ring. Ring R is said to be **weakly presimplifiable**, if for any $a, b \in R$ with $a = ab$, then $a = 0$ or b is regular.

Example 8. Let $R = F[X, Y, Z]/(X - XYZ) = F[x, y, z]$ with F is a field and X, Y, Z indeterminates over F. Here $Z(R) = (x) \cup (1 - yz)$. By Theorem 5, $(x), (1 - yz) \subseteq Z(R), (x) + (1 - yz) \neq R$, such that R is weakly presimplifiable ring.

We next show the properties that equivalent to the weakly presimplifiable ring [4].

Theorem 12. *For a commutative ring R with identity, the following conditions are equivalent.*

(1). $\forall a, b \in R$, $a \sim b$, implies $a \cong_r b$.
(2). $\forall a, b \in R$, $a \approx_r b$, implies $a \cong_r b$.
(3). $\forall a, b \in R$, $a \approx b$, implies $a \cong_r b$.
(4). $\forall a \in R$, $a \cong_r a$.
(5). R is weakly presimplifiable ring.
(6). $Z(R) \subseteq 1 - reg(R)$.
(7). For (prime) ideals $P, Q \subseteq Z(R)$, $P + Q \neq R$.
(8). $\forall a, b \in Z(R)$, $(a, b) \neq R$.
(9). $\forall a \in R$, a or $a - 1$ is regular.
(10). $\forall 0 \neq r \in R$, $sRr = Rr$, implies s is regular.

Proof. (1) ⇒ (2) Let $a, b \in R$, $a \sim b$. Suppose that $a \approx_r b$. Since $a \sim b$ and $a = rb$ for some regular element $r \in R$, then $a \cong_r b$.
(2) ⇒ (3) Let $a, b \in R$ with $a \approx_r b$. Suppose that $a \approx b$. Since $a = rb$ and $b = sa$ for some $r, s \in R$, such that $a \sim b$. Hence $a \sim b$ and $a = ub$ for $u \in U(R) \subseteq R$, then $a \cong_r b$.
(3) ⇒ (4) Let $a \in R$. Since $a = ua$ for some $u \in U(R)$, such that $a \approx a$. Thus $a \cong_r a$.
(4) ⇒ (5) Let $a \in R$. Assume that $a = ba$, then $a \cong_r a$ for $b \in R$. Since $a = 0$ or b is regular, such that R is a weakly presimplifiable ring.
(5) ⇒ (6) Let $z \in Z(R)$, so there is $0 \neq x \in R$ with $xz = 0$. Then $x = x - xz = x(1 - z)$, so $(1 - z) \in R$ is regular. Then $z = 1 - (1 - z) \in 1 - reg(R)$. Thus $Z(R) \subseteq 1 - reg(R)$.
(6) ⇒ (7) Suppose that $P + Q = R$, there exist $p \in P$ and $q \in Q$ with $p + q = 1$. Now $q = 1 - r$ with $r \in reg(R)$ and hence $p + q = 1 \Leftrightarrow 1 - p = q = 1 - r$. So, $p = r$ is regular, a contradiction. Thus, $P + Q \neq R$.
(7) ⇒ (8) Let $a, b \in Z(R)$. Since $ar = 0$ and $br = 0$, so $(a + b)r = 0$. Then $(a + b) \in Z(R)$ is regular. Thus $(a, b) \neq R$.
(8) ⇒ (9) Let $a \in R$. Assume that $a \in Z(R)$. Then $(a + 0)$ is regular in $Z(R)$. So $a \in R$ is regular.
(9) ⇒ (10) Let r be nonzero element in R is regular. Suppose that $sRr = Rr$. Since $r \in R$ is regular, so $s = r$. Hence $s \in R$ is regular.
(10) ⇒ (1) Let $a, b \in R$. Suppose that $a \sim b$. Since r and s are regular, then $a \cong_r b$. □

Proposition 2. [4] *Weakly presimplifiable R is strongly regular associate ring.*

Proof. (⇒) Let R be weakly presimplifiable ring. We now show that R is strongly regular associate ring. Assume that $a \sim b$. Then $a = rb$ and $b = sa$ for some $r, s \in R$, hence $a = r(sa)$. Since R is weakly presimplifiable ring, then $rs \in R$ is regular. Therefore, r and s are regular in R. Thus $a \approx_r b$.
(⇐) Let R be strongly regular associate ring. Suppose that $a = ba$ for some $a, b \in R$. We must show that R is weakly presimplifiable ring. Since $(a) = (a)$, then $a \sim a$. Hence $a \approx_r a$ since R strongly regular associate ring. If a is nonzero element in R, then $a = ba$ for some $b \in R$. So $b \in R$ is regular. Thus R is weakly presimplifiable ring. □

The following is given the theorem regarding subring in a weak presimplifible ring.

Theorem 13. *Let R be a commutative ring with identity is weakly presimplifiable ring. Suppose a nonempty set S in R. Then S is subring of R, if S is weakly presimplifiable ring.*
Proof. First, $S \neq \emptyset$, *since* $1 \in S$ *is regular, so* $1 \cdot s = s \in S$. Then, $s \cdot t = s \in S\ \forall s, t \in S$. Since (-1) is regular in S, then $s - t = t(-1) = t \in S$. Thus, S be subring over R. □

Furthermore, it will be given theorems regarding the relationship of weakly presimplifiable ring with polynomial ring $R[X]$ and formal power series ring $R[[X]]$.

Theorem 14. [4] *Let R be a commutative ring and $\{X_i\}$ a nonempty set of indeterminates over R. Formal power series $R[\{X_i\}]$ is weakly presimplifiable ring if and only if R is weakly presimplifiable ring.*

Proof. ($\Rightarrow$) Let $x = xy$, $x, y \in R$. Since $R \subseteq R[\{X_i\}]$, then $x = xy$ also in $R[\{X_i\}]$. So $x = 0$ or $y \in reg(R[\{X_i\}]) \cap R = reg(R)$. Then $x = 0$ or $y \in reg(R)$ also in ring R. Thus R is weakly presimplifiable ring.
($\Leftarrow$)Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$. If coefficient $a_0 \in R$ is regular, then $f(X) \in R[X]$ is regular. If coefficient $a_0 \in R$ is not regular, but R is weakly presimplifiable ring, then $a_0 - 1$ is regular. By Theorem 12, such that $f - 1$ is regular. Thus, $R[X]$ is weakly presimplifiable ring. □

Theorem 15. [4] *Let R be a commutative ring. Formal power series $R\left[[X_1, \dots, X_n]\right]$ is weakly presimplifiable ring if and only if R is weakly presimplifiable ring.*

Proof. ($\Rightarrow$) Suppose that $x = xy$, $x, y \in R$. Then $x = xy$ in $R\left[[X_1, \dots, X_n]\right]$, so $x = 0$ or y is regular in $R\left[[X_1, \dots, X_n]\right]$ and hence in R.
($\Leftarrow$) Let $f \in Z\left(R\left[[X_1, \dots, X_n]\right]\right)$. Then the constant term $a \in f$ lies in $Z(R)$. So $1 - a \in reg(R)$. Then the constant term of $1 - f$ is regular. Thus $1 - f$ is regular.

4. CONCLUSIONS

The conclusions that the author can draw are as follows.

1) Ring R is said to be presimplifiable ring if and only if R is a very strongly associate ring.
2) Ring R is said to be weakly presimplifiable ring if and only if R is strongly regular associate ring.
3) If R be weakly presimplifiable ring, then $R[X]$ and $R[[X]]$ be weakly presimplifiable rings. However, if R be presimplifiable ring, then $R[[X]]$ be presimplifiable ring but $R[X]$ is not.

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