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ON THE GIRTH, INDEPENDENCE NUMBER, AND WIENER INDEX OF COPRIME GRAPH OF DIHEDRAL GROUP

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ABSTRACT

The coprime graph of a finite group G, denoted by Γ_G, is a graph with vertex set G such that two distinct vertices x and y are adjacent if and only if their orders are coprime, i.e., $\gcd(|x|,|y|) = 1$ where $|x|$ is the order of x. In this paper, we complete the form of the coprime graph of a dihedral group that was given by previous research and it has been proved that $girth(\Gamma_{D_{2n}}) = \infty$ if $n = 2^k$, for some $k \geq 2$ and $girth(\Gamma_{D_{2n}}) = 3$ if $n \neq 2^k$. Moreover, we prove that if n is even, then the independence number of $\Gamma_{D_{2n}}$ is $\alpha(\Gamma_{D_{2n}}) = 2n - m$, where m is the greatest odd divisor of n and if n is odd, then the independence number of $\Gamma_{D_{2n}}$ is $\alpha(\Gamma_{D_{2n}}) = n$. Furthermore, the Wiener index of coprime graph of dihedral group has been stated here.

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1. INTRODUCTION

Algebraic graph theory is one of mathematic branches that has been developed. It combines two fields in mathematics, i.e., graph theory and algebraic structure (group theory, ring theory, etc.). We use the standard terminology of graphs (e.g., see [1], [2], [3]) and groups (e.g., see [4], [5]). Some examples of algebraic graphs are the commuting graph [6], cyclic graph [7], prime graph [8], and non-coprime graph [9].

In 2014, Ma, et al. [10] defined the coprime graph of finite groups. The coprime graph of a finite group G is a graph with vertex set G such that two distinct vertices x and y are adjacent if and only if $\gcd(|x|,|y|)=1$ where $|x|$ is the order of x. The coprime graph of group G is denoted by Γ_G.

Research related to coprime graphs is growing rapidly. The necessary and sufficient conditions for a coprime graph, which is a complete k −partite graph or planar graph, were explained by Dorbidi in [11]. In 2020, Alimon, et al. [12] examined the Szeged and Wiener index of the coprime graph of dihedral group D_{2n} for several n. Then Gazir, et al. [13] described some properties of a coprime graph of a dihedral group D_{2n} where n is a prime power. In 2021, Syarifudin, et al. [14] continued the research conducted by [13] observing the diameter, radius, and girth of the coprime graph of a dihedral group D_{2n} for n in general. However, there are still unexplained conditions of n for girth. In the same year, Hamm and Way [15] explained the independence number of the coprime graph of dihedral groups D_{2n} for $n=2^l m$. The other conditions of n have not been explained.

In this paper, we will provide some properties for the form, girth, independence number, and the Wiener index of the coprime graph of dihedral group D_{2n} that have not been given in [12], [14] and [15].

2. RESEARCH METHODS

This research is based on theoretical literature studies to gain new knowledge from existing sciences. First, we determine the research topic by studying topics related to coprime graphs of finite groups and find research gaps. Then, we divide the problem into several cases and make conjectures using examples for each case. Next, we prove the conjecture by deductive proof.

3. RESULTS AND DISCUSSION

This research discusses the girth, independence number, and Wiener index of the coprime graph of the dihedral group. The definition of a dihedral group is as follows.

Definition 1. [4] Let $n\in\mathbb{N}$ and $n\geq 3$. A dihedral group of order $2n$, denoted by D_{2n}, is a group generated by $a,b\in G$ such that

$$D_{2n}=\langle a,b \mid a^n=b^2=e, ab=ba^{-1}\rangle.$$

The dihedral group can also be written as $D_{2n}=\{e,a,a^2,\dots,a^{n-1},b,ab,a^2b,\dots,a^{n-1}b\}$. The coprime graph $\Gamma_{D_{2n}}$ of a dihedral group D_{2n}, has vertex set

$$V(\Gamma_{D_{2n}})=D_{2n}=\{e,a,a^2,\dots,a^{n-1},b,ab,a^2b,\dots,a^{n-1}b\}.$$

Gazir, et al. [13] and Syarifudin, et al. [14] gave some theorems about the form of the coprime graph of dihedral group D_{2n}. These theorems are as follows.

Theorem 1. [13] *Let n be an odd prime number, then the coprime graph $\Gamma_{D_{2n}}$ is a complete tripartite graph.*

Theorem 2. [13] *Let $n=2^k$, for some $k\in\mathbb{N}$, then the coprime graph $\Gamma_{D_{2n}}$ is a complete bipartite graph.*

Theorem 3. [13] *Let $n=p^k$, $p\neq 2$, and p is prime number, for some $k\in\mathbb{N}$, then the coprime graph $\Gamma_{D_{2n}}$ is a complete tripartite graph.*

Theorem 4. [14] *Let $n=p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$, where $1\leq i\leq m$, p_i are distinct prime number, and $p_i\neq 2$, then the coprime graph of D_{2n} is $(m+2)-$ partite.*

A more specific form of **Theorem 2** is described in the following proposition.

Proposition 1. *Let $n = 2^k$, for some $k \in \mathbb{N}$, $k \geq 2$. The coprime graph of D_{2n} is isomorphic to star graph S_{2n-1}.*

Proof. Let D_{2n} be a dihedral group with $n = 2^k$, for some $k \in \mathbb{N}$, $k \geq 2$. It is clear that $|e| = 1$. Since $n = 2^k$ with $k \in \mathbb{N}$, then for all $i = 1,2,\dots,n-1$, $|a^i| = 2^l$, for some $l \leq k$. On the other hand, $|b| = |ab| = |a^2b| = \cdots = |a^{n-1}b| = 2$. Hence, for all $x, y \in V(\Gamma_{D_{2n}}) \setminus \{e\}$, $gcd(|x|,|e|) = 1$ and $gcd(|x|,|y|) = 2^l \neq 1$ for some $l \leq k$. Then $E(\Gamma_{D_{2n}}) = \left\{ ex \mid x \in V(\Gamma_{D_{2n}}) \setminus \{e\} \right\}$. Thus, the coprime graph of D_{2n} with $n = 2^k$, for $k \geq 2$ is isomorphic to the star graph S_{2n-1}. ∎

However, there is still an unexplored condition of n for the form of the coprime graph of the dihedral group D_{2n}, so the following proposition is obtained.

Proposition 2. *Let $n \geq 3$ be any positive integer. The coprime graph $\Gamma_{D_{2n}}$ is $(m+2)$ −partite if n odd and it is $(m+1)$ −partite if n even where m is the number of distinct prime factors of n.*

Proof. Let $n \geq 3$ be any positive integer. Suppose $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, where for each $i = 1,2,\dots,m$, $k_i \in N$ and p_i are distinct prime numbers. If n is odd, according to **Theorem 4**, the coprime graph $\Gamma_{D_{2n}}$ is $(m+2)-$ partite. If n is even, there is $i \in \{1,2,\dots,m\}$ such that $p_i = 2$. Then define the set A_t as the set of all elements D_{2n} with order t. Take partitions $V_1, V_2, \dots, V_{m+1}$ on $V(\Gamma_{D_{2n}})$ where $V_1 = A_1$,

$$V_2 = A_{p_1} \cup \bigcup_{i=1}^{k_1} A_{p_1^i} \cup \bigcup_{t \in \mathbb{N}} A_{tp_1},$$

$$V_3 = A_{p_2} \cup \bigcup_{i=1}^{k_2} A_{p_2^i} \cup \bigcup_{t \in \mathbb{N}, p_1 \nmid t} A_{tp_2},$$

$$V_4 = A_{p_3} \cup \bigcup_{i=1}^{k_3} A_{p_3^i} \cup \bigcup_{t \in \mathbb{N}, p_1, p_2 \nmid t} A_{tp_3},$$

$$\vdots$$

$$V_{m+1} = A_{p_m} \bigcup_{i=1}^{k_m} A_{p_m^i}.$$

For each $i = 1, 2, \dots, m$, note that for every $x, y \in V_i$ then $p_i \mid |x|$ and $p_i \mid |y|$ so that $gcd(|x|,|y|) \neq 1$. Thus, the coprime graph $\Gamma_{D_{2n}}$ is $(m+1)$ −partite. Furthermore, for $m \geq 2$ we have $\left|a^{p_3^{k_3} \dots p_m^{k_m}}\right| = p_1^{k_1} p_2^{k_2}$ and $\left|a^{p_1^{k_1} p_3^{k_3} \dots p_m^{k_m}}\right| = p_2^{k_2}$ so that $a^{p_3^{k_3} \dots p_m^{k_m}} \in V_2$ and $a^{p_1^{k_1} p_3^{k_3} \dots p_m^{k_m}} \in V_3$. Then

$$gcd\left(\left|a^{p_3^{k_3} \dots p_m^{k_m}}\right|, \left|a^{p_1^{k_1} p_3^{k_3} \dots p_m^{k_m}}\right|\right) = gcd(p_1^{k_1} p_2^{k_2}, p_2^{k_2}) = p_2^{k_2} \neq 1.$$

Hence, $a^{p_3^{k_3} \dots p_m^{k_m}}$ is not adjacent to $a^{p_1^{k_1} p_3^{k_3} \dots p_m^{k_m}}$. In other words, the coprime graph $\Gamma_{D_{2n}}$ is not complete $(m+1)$ −partite. ∎

Note that, the girth of graph Γ, denoted by $girth(\Gamma)$, is defined as the length of the shortest cycle in Γ. The following theorem given in [14] is about the girth of the coprime graph D_{2n}.

Theorem 5. [14] *Let D_{2n} be dihedral group. If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ and $2 \nmid n$, then girth of $\Gamma_{D_{2n}}$ is 3.*

The following theorem completes the **Theorem 5**.

Theorem 6. *Let $n \geq 3$ be any positive integer. If $n = 2^k$, for some $k \geq 2$, then $girth(\Gamma_{D_{2n}}) = \infty$. If $n \neq 2^k$, then $girth(\Gamma_{D_{2n}}) = 3$.*

Proof. Let $n \geq 3$ be any positive integer. If $n = 2^k$, for some $k \geq 2$, by **Proposition 1**, $\Gamma_{D_{2n}} \cong S_{2n-1}$. Then, $girth(\Gamma_{D_{2n}}) = \infty$. For $n \neq 2^k$, suppose $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, where for every $i = 1, 2, \dots, m$, $k_i \in \mathbb{N}$ and p_i

are distinct prime numbers. If n is odd, by **Theorem 5**, $girth(\Gamma_{D_{2n}}) = 3$. If n is even, WLOG, suppose $p_1 = 2$. Note that $|e| = 1$, $\left|a^{p_1^{k_1}}\right| = p_2^{k_2} \dots p_m^{k_m}$, and $|b| = 2$. Clearly that $2 \nmid p_2^{k_2} \dots p_m^{k_m}$, so that we have $gcd(|e|, |a|) = 1$, $gcd(|a|, |b|) = 1$, and $gcd(|b|, |e|) = 1$. It means that in $\Gamma_{D_{2n}}$ there is cycle $e - a - b - e$. Hence $girth(\Gamma_{D_{2n}}) = 3$. ■

Next, an independent set of graph Γ is a subset S of $V(\Gamma)$ such that every two vertices of S are not adjacent in Γ. The independence number $\alpha(\Gamma)$ is the size of the maximum independent set of Γ. The independence number of the coprime graph $\Gamma_{D_{2n}}$ for even number n is explained in Theorem 4.1 in **[15]** as follows.

Theorem 7. [15] *Let $G = D_{2n}$ with $n = 2^l m$, where m is odd and l is an integer. Then $\alpha\left(\Gamma_{D_{2n}}\right) = 2n - m$.*

The following theorem completes the **Theorem 7**.

Theorem 8. *Let $n \geq 3$ be any positive integer. If n is even, then $\alpha\left(\Gamma_{D_{2n}}\right) = 2n - m$, where m is the greatest odd divisor of n. If n is odd, then $\alpha(\Gamma_{D_{2n}}) = n$.*

Proof. Let $n \geq 3$ be any positive integer. If n is even, suppose $n = 2^l m$, where m is odd and l is an integer. By **Theorem 7**, $\alpha\left(\Gamma_{D_{2n}}\right) = 2n - m$. If n is odd, suppose $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, where for every $i = 1, 2, \dots, m$, $k_i \in \mathbb{N}$ and p_i are distinct odd prime numbers. Note that $|b| = |ab| = |a^2 b| = \cdots = |a^{n-1} b| = 2$, then $V = \{b, ab, a^2 b, \dots, a^{n-1} b\}$ is an independent set with cardinality n. Assuming V is not a maximum independent set, it means that there is an independent set U with $|U| > n$. Then U contains elements of the form $a^i b$ and a^j with $i, j \in \{1, 2, \dots, n-1\}$. Since n is odd, $|a^j|$ is also odd, so $gcd\left(|a^i b|, |a^j|\right) = 1$. Therefore, U is not an independent set, there is a contradiction. In other words, V is the maximum independent set, then $\alpha(\Gamma_{D_{2n}}) = n$. ■

Then, the Wiener index of graph Γ, denoted by $W(\Gamma)$, is defined as $W(\Gamma) = \sum_{u,v \in V(\Gamma)} d(u, v)$, where $d(u, v)$ is the distance between u and v. Alimon, et al. **[12]** had explained that the Wiener index of a coprime graph of dihedral group D_{2n} for n is odd prime and $n = 2^k$ for some $k \in \mathbb{N}$.

Theorem 9. [12] *Let G be the dihedral group of order $2n$, where $n \geq 3$ and Γ_G is coprime graph of G. Then, if n is an odd prime, the Wiener index of coprime graph for D_{2n} is as follows:*

$$W(\Gamma_G) = 3n^2 - 3n + 1.$$

Theorem 10. [12] *Let G be dihedral group of order $2n$, where $n \geq 3$ and Γ_G is coprime graph of G. Then, if $n = 2^k$, where $k \in \mathbb{Z}^+$, the Wiener index of coprime graph for D_{2n} is as follows:*

$$W(\Gamma_G) = (2n - 1)^2.$$

The following theorems will explain the Wiener index of a coprime graph of dihedral group D_{2n} for some other n.

Theorem 11. *Let $n = p^k$, where p is an odd prime number and $k \in \mathbb{N}$. Then $W\left(\Gamma_{D_{2n}}\right) = 3n^2 - 3n + 1$.*

Proof. Let $n = p^k$, where p is an odd prime number and $k \in \mathbb{N}$. By **Theorem 3**, the coprime graph $\Gamma_{D_{2n}}$ is a complete tripartite graph with partitions $V_1 = \{e\}$, $V_2 = \{a, a^2, \dots, a^{n-1}\}$, and $V_3 = \{b, ab, a^2 b, \dots, a^{n-1} b\}$. From that partitions we get

$$\sum_{v \in V(\Gamma_{D_{2n}}) \setminus V_1} d(e, v) = 1(2n - 1) = 2n - 1,$$

$$\sum_{u,v \in V_2} d(u, v) = 2\binom{n-1}{2} = (n-1)(n-2),$$

$$\sum_{u,v \in V_3} d(u, v) = 2\binom{n}{2} = n(n-1),$$

$$\sum_{u\in V_2, v\in V_3} d(u,v) = 1\big(n(n-1)\big) = n(n-1).$$

As a result, the Wiener index of the coprime graph $\Gamma_{D_{2n}}$ is

$$\begin{aligned} W\left(\Gamma_{D_{2n}}\right) &= \sum_{v\in V(\Gamma_{D_{2n}})\setminus V_1} d(e,v) + \sum_{u,v\in V_2} d(u,v) + \sum_{u,v\in V_3} d(u,v) + \sum_{u\in V_2, v\in V_3} d(u,v) \\ &= (2n-1) + (n-1)(n-2) + n(n-1) + n(n-1) \\ &= 2(n-1) + 1 + (n-1)(n-2) + n(n-1) + n(n-1) \\ &= (n-1)(2+n-2+n+n) + 1 \\ &= (n-1)3n + 1 \\ &= 3n^2 - 3n + 1. \ \blacksquare \end{aligned}$$

Theorem 12. *Let $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, where for every $i = 1, 2, \dots, m$, $k_i \in \mathbb{N}$ and p_i are distinct odd prime numbers. Then the Wiener index of the coprime graph $\Gamma_{D_{2n}}$ is*

$$\begin{aligned} W\left(\Gamma_{D_{2n}}\right) = {} & 2n^2 - 1 + \sum_{i=1}^{m} (p_i^{k_i} - 1)(p_i^{k_i} - 2) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (p_i^{k_i} - 1)\left(p_j^{k_j} - 1\right) + \sum_{t\in D} 2\binom{\varphi(t)}{2} \\ & + \sum_{t,s\in D, \gcd(t,s)=1} \varphi(t)\varphi(s) + \sum_{t,s\in D, \gcd(t,s)\neq 1} 2\varphi(t)\varphi(s) + \sum_{i=1}^{m} \sum_{t\in D, p_i \nmid t} (p_i^{k_i} - 1)\varphi(t) \\ & + \sum_{i=1}^{m} \sum_{t\in D, p_i \mid t} 2(p_i^{k_i} - 1)\varphi(t), \end{aligned}$$

where D is the set of all factors of n that are not primes power and $\varphi(t)$ is the Euler function of t.

Proof. Let $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, where for every $i = 1, 2, \dots, m$, $k_i \in \mathbb{N}$ and p_i are distinct odd prime numbers. Suppose that C is the set of all factors of n. We form a partition P on $V\left(\Gamma_{D_{2n}}\right)$ with

$$P = \{A_t \mid A_t \text{ is the set of all elements in } D_{2n} \text{ with order } t, t \in C\}.$$

For all $i = 1, 2, \dots, m$ let $B_{p_i} = A_{p_i} \cup A_{p_i^2} \cup \dots \cup A_{p_i^{k_i}}$. Suppose that D is the set of all factors of n that are not primes power. Therefore, there are some conditions as follows.

1) For all $v \in V\left(\Gamma_{D_{2n}}\right) \setminus A_1$, $d(e,v) = 1$. Then $\sum_{v\in V(\Gamma_{D_{2n}})\setminus A_1} d(e,v) = 1(2n-1) = 2n-1$.
2) For all $u, v \in A_2$, $d(u,v) = 2$. Then $\sum_{u,v\in A_2} d(u,v) = 2\binom{n}{2} = n(n-1)$.
3) For all $u \in A_2$ and $v \in V\left(\Gamma_{D_{2n}}\right) \setminus (A_1 \cup A_2)$, $d(u,v) = 1$. Then
$$\sum_{u\in A_2, v\in V(\Gamma_{D_{2n}})\setminus(A_1\cup A_2)} d(u,v) = 1\big(n(n-1)\big) = n(n-1).$$
4) For all $i = 1,2, \dots, m$. For all $u, v \in B_{p_i}$, $d(u,v) = 2$. Then
$$\sum_{i=1}^{m} \sum_{u,v\in B_{p_i}} d(u,v) = \sum_{i=1}^{m} 2\binom{p_i^{k_i} - 1}{2} = \sum_{i=1}^{m} (p_i^{k_i} - 1)(p_i^{k_i} - 2).$$
5) For all $i, j = 1,2, \dots, m$ with $i \neq j$. For all $u \in B_{p_i}$ and $v \in B_{p_j}$, $d(u,v) = 1$. Then
$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \sum_{u\in B_{p_i}, v\in B_{p_j}} d(u,v) = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (p_i^{k_i} - 1)\left(p_j^{k_j} - 1\right).$$
6) For all $t \in D$. For all $u, v \in A_t$, $d(u,v) = 2$. Then

$$\sum_{t\in D}\sum_{u,v\in A_t} d(u,v) = \sum_{t\in D} 2\binom{\varphi(t)}{2}.$$

7) For all $t,s \in D$. If $\gcd(t,s)=1$, then for all $u \in A_t$ and $v \in A_s$, $d(u,v)=1$. Thus

$$\sum_{t,s\in D,\gcd(t,s)=1}\ \sum_{u\in A_t, v\in A_s} d(u,v) = \sum_{t,s\in D,\gcd(t,s)=1} \varphi(t)\varphi(s).$$

If $\gcd(t,s) \neq 1$, then for all $u \in A_t$ and $v \in A_s$, $d(u,v)=2$. Thus

$$\sum_{t,s\in D,\gcd(t,s)\neq 1}\ \sum_{u\in A_t, v\in A_s} d(u,v) = \sum_{t,s\in D,\gcd(t,s)=1} 2\varphi(t)\varphi(s).$$

8) For all $i=1,2,\dots,m$ and $t \in D$. If $p_i \nmid t$, then for all $u \in B_{p_i}$ and $v \in A_t$, $d(u,v)=1$. Thus

$$\sum_{i=1}^{m}\sum_{t\in D, p_i\nmid t}\ \sum_{u\in B_{p_i}, v\in A_t} d(u,v) = \sum_{i=1}^{m}\sum_{t\in D, p_i\nmid t} (p_i^{k_i}-1)\varphi(t).$$

Based on all these conditions, we obtain the Wiener index of the coprime graph $\Gamma_{D_{2n}}$ as follows

$$W(\Gamma_{D_{2n}}) = 2n^2 - 1 + \sum_{i=1}^{m}(p_i^{k_i}-1)(p_i^{k_i}-2) + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(p_i^{k_i}-1)\left(p_j^{k_j}-1\right) + \sum_{t\in D} 2\binom{\varphi(t)}{2}$$
$$+ \sum_{t,s\in D,\gcd(t,s)=1}\varphi(t)\varphi(s) + \sum_{t,s\in D,\gcd(t,s)\neq 1} 2\varphi(t)\varphi(s) + \sum_{i=1}^{m}\sum_{t\in D, p_i\nmid t}(p_i^{k_i}-1)\varphi(t)$$
$$+ \sum_{i=1}^{m}\sum_{t\in D, p_i\mid t} 2(p_i^{k_i}-1)\varphi(t)\,. \quad \blacksquare$$

Theorem 13. *Let $n = 2^k p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, where for every $i = 1,2,\dots,m$, $k, k_i \in \mathbb{N}$ and p_i are distinct odd prime numbers. Then the Wiener index of the coprime graph $\Gamma_{D_{2n}}$ is*

$$W(\Gamma_{D_{2n}}) = 2n - 1 + (n+2^k-1)(n+2^k-2) + \sum_{i=1}^{m}(n+2^k-1)\left(p_i^{k_i}-1\right)$$
$$+ \sum_{t\in D, 2\nmid t}(n+2^k-1)\varphi(t) + \sum_{t\in D, 2\mid t} 2(n+2^k-1)\varphi(t) + \sum_{i=1}^{m}(p_i^{k_i}-1)(p_i^{k_i}-2)$$
$$+ \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(p_i^{k_i}-1)\left(p_j^{k_j}-1\right) + \sum_{t\in D} 2\binom{\varphi(t)}{2} + \sum_{t,s\in D,\gcd(t,s)=1}\varphi(t)\varphi(s)$$
$$+ \sum_{t,s\in D,\gcd(t,s)\neq 1} 2\varphi(t)\varphi(s) + \sum_{i=1}^{m}\sum_{t\in D, p_i\nmid t}(p_i^{k_i}-1)\varphi(t) + \sum_{i=1}^{m}\sum_{t\in D, p_i\mid t} 2(p_i^{k_i}-1)\varphi(t)$$

where D is the set of all factors of n that are not primes power and $\varphi(t)$ is the Euler function of t.

Proof. Let $n = 2^k p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, where for every $i = 1,2,\dots,m$, $k, k_i \in \mathbb{N}$ and p_i are distinct odd prime numbers. Suppose that C is the set of all factors of n. We form a partition P on $V(\Gamma_{D_{2n}})$ with

$$P = \{A_t \mid A_t \text{ is the set of all elements in } D_{2n} \text{ with order } t, t \in C\}.$$

For all $i = 1,2,\dots,m$ let $B_{p_i} = A_{p_i} \cup A_{p_i^2} \cup \dots \cup A_{p_i^{k_i}}$. Suppose that D is the set of all factors of n that are not primes power. Therefore, there are some conditions as follows.

1) For all $v \in V(\Gamma_{D_{2n}}) \setminus A_1$, $d(e,v)=1$. Then $\sum_{v\in V(\Gamma_{D_{2n}})\setminus A_1} d(e,v) = 1(2n-1) = 2n-1$.

2) For all $u,v \in B_2$, $d(u,v)=2$. Then $\sum_{u,v\in B_2} d(u,v) = 2\binom{n+2^k-1}{2} = (n+2^k-1)(n+2^k-2)$.

3) For all $i=1,2,\dots,m$. For all $u \in B_2$ and $v \in B_{p_i}$, $d(u,v)=1$. Then

$$\sum_{u \in B_2, v \in B_{p_i}} d(u,v) = \sum_{i=1}^{m} (n + 2^k - 1)\left(p_i^{k_i} - 1\right).$$

4) For all $t \in D$. If t is odd, then for all $u \in B_2$ and $v \in A_t$, $d(u,v) = 1$. Thus

$$\sum_{t \in D, 2 \nmid t} \sum_{u \in B_2, v \in A_t} d(u,v) = \sum_{t \in D, 2 \nmid t} (n + 2^k - 1)\varphi(t).$$

If t is even, then for all $u \in B_2$ and $v \in A_t$, $d(u,v) = 2$. Thus

$$\sum_{t \in D, 2 | t} \sum_{u \in B_2, v \in A_t} d(u,v) = \sum_{t \in D, 2 | t} (n + 2^k - 1)\varphi(t).$$

5) For all $i = 1,2, \dots, m$. For all $u, v \in B_{p_i}$, $d(u,v) = 2$. Then

$$\sum_{i=1}^{m} \sum_{u,v \in B_{p_i}} d(u,v) = \sum_{i=1}^{m} 2\binom{p_i^{k_i} - 1}{2} = \sum_{i=1}^{m} (p_i^{k_i} - 1)(p_i^{k_i} - 2).$$

6) For all $i, j = 1,2, \dots, m$ with $i \neq j$. For all $u \in B_{p_i}$ and $v \in B_{p_j}$, $d(u,v) = 1$. Then

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \sum_{u \in B_{p_i}, v \in B_{p_j}} d(u,v) = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (p_i^{k_i} - 1)\left(p_j^{k_j} - 1\right).$$

7) For all $t \in D$. For all $u, v \in A_t$, $d(u,v) = 2$. Then

$$\sum_{t \in D} \sum_{u,v \in A_t} d(u,v) = \sum_{t \in D} 2\binom{\varphi(t)}{2}.$$

8) For all $t, s \in D$. If $\gcd(t,s) = 1$, then for all $u \in A_t$ and $v \in A_s$, $d(u,v) = 1$. Thus

$$\sum_{t,s \in D, \gcd(t,s)=1} \sum_{u \in A_t, v \in A_s} d(u,v) = \sum_{t,s \in D, \gcd(t,s)=1} \varphi(t)\varphi(s).$$

If $\gcd(t,s) \neq 1$, then for all $u \in A_t$ and $v \in A_s$, $d(u,v) = 2$. Thus

$$\sum_{t,s \in D, \gcd(t,s) \neq 1} \sum_{u \in A_t, v \in A_s} d(u,v) = \sum_{t,s \in D, \gcd(t,s)=1} 2\varphi(t)\varphi(s).$$

9) For all $i = 1,2, \dots, m$ and $t \in D$. If $p_i \nmid t$, then for all $u \in B_{p_i}$ and $v \in A_t$, $d(u,v) = 1$. Thus

$$\sum_{i=1}^{m} \sum_{t \in D, p_i \nmid t} \sum_{u \in B_{p_i}, v \in A_t} d(u,v) = \sum_{i=1}^{m} \sum_{t \in D, p_i \nmid t} (p_i^{k_i} - 1)\varphi(t).$$

If $p_i \mid t$, then for all $u \in B_{p_i}$ and $v \in A_t$, $d(u,v) = 2$. Thus

$$\sum_{i=1}^{m} \sum_{t \in D, p_i | t} \sum_{u \in B_{p_i}, v \in A_t} d(u,v) = \sum_{i=1}^{m} \sum_{t \in D, p_i | t} 2(p_i^{k_i} - 1)\varphi(t)$$

Based on all these conditions, we conclude that the Wiener index of the coprime graph $\Gamma_{D_{2n}}$ is

$$
\begin{aligned}
W(\Gamma_{D_{2n}}) = {} & 2n - 1 + (n + 2^k - 1)(n + 2^k - 2) + \sum_{i=1}^{m} (n + 2^k - 1)\left(p_i^{k_i} - 1\right) \\
& + \sum_{t \in D, 2 \nmid t} (n + 2^k - 1)\varphi(t) + \sum_{t \in D, 2 \mid t} 2(n + 2^k - 1)\varphi(t) + \sum_{i=1}^{m} (p_i^{k_i} - 1)(p_i^{k_i} - 2) \\
& + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (p_i^{k_i} - 1)\left(p_j^{k_j} - 1\right) + \sum_{t \in D} 2\binom{\varphi(t)}{2} + \sum_{t,s \in D, \gcd(t,s)=1} \varphi(t)\varphi(s) \\
& + \sum_{t,s \in D, \gcd(t,s) \neq 1} 2\varphi(t)\varphi(s) + \sum_{i=1}^{m} \sum_{t \in D, p_i \nmid t} (p_i^{k_i} - 1)\varphi(t) \\
& + \sum_{i=1}^{m} \sum_{t \in D, p_i \mid t} 2(p_i^{k_i} - 1)\varphi(t). \quad \blacksquare
\end{aligned}
$$

4. CONCLUSIONS

In this study we complete the properties of girth, independence number, and Wiener index of a coprime graph of a dihedral group that was given by the previous research in **[14]**, **[15]**, and **[12]**.

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