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# SOME FUNDAMENTAL PROPERTIES OF HEAPS

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## ABSTRACT

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*Heap is defined to be a non-empty set $H$ with ternary operation $[-,-,-]: H \times H \times H \to H$ satisfying associativity, that is $\left([[a,b,c],d,e] = [a,b,[c,d,e]]\right)$ for every $a,b,c,d,e \in H$ and satisfying Mal'cev identity, that is $[a,b,b] = b = [b,b,a]$ for all $a,b \in H$. There is a connection between heaps and groups. From a given heap, we can construct some groups and vice versa. The binary operation of groups can be built by choosing any fixed element $e$ of heap $H$ and is defined by $x \cdot_e y = [x,e,y]$ for any $x,y \in H$. Otherwise, for given a binary operation of group $G$, we can make a ternary operation defined by $[x,y,z] = xy^{-1}z$ for every $x,y,z \in G$. On heaps, there are some notions which are inspired by groups, such as sub-heaps, normal sub-heaps, quotient heaps, and heap morphisms. On this study, we will associate sub-heaps and corresponding subgroups and discuss some properties of heap morphisms.*



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## 1. INTRODUCTION

A semigroup is a non-empty set equipped with closed and associative binary operation. The set of binary relation on set $A$ is a semigroup under composition. However, if we have distinct sets $A$ and $B$, then the set of binary relation from $A$ to $B$ cannot be composed. To fix this problem, it is embedded an inverse of binary relation between them. This defines a ternary operation. Furthermore, the set of binary relation from $A$ to $B$ with that ternary operation is a semiheap. A semiheap $K$ is nonempty set with closed and pseudo-associative ternary operation $\left([[k_1,k_2,k_3],k_4,k_5] = [k_1,[k_4,k_3,k_2],k_5] = [k_1,k_2,[k_3,k_4,k_5]],\right.$ for any $\left.k_1,k_2,k_3,k_4,k_5 \in K\right)$. A semiheap $K$ which satisfies Mal'cev identity $[a,b,b] = a = [b,b,a]$ for any $a,b \in K)$ is called a heap [1].

Heap was introduced in [2] and [3] for the first time. Now, heaps theory have been studied widely in many areas. In mathematics, the studies of heaps have been on advanced level, such as category of heaps [4], modules theory [5], near heaps [6], and generalized heaps [1].

Heaps can be viewed as a generalization of groups by forgetting the identity element. According to [3] and [7] we can construct heaps from groups and vice versa. However, there are only few studies of heaps which were associated to group properties. Those properties are the relation between homomorphism of groups and morphisms of heaps and the relation between normal subgroups and normal sub-heaps [8]. The former researches of heaps bring us to study and observe further fundamental properties of heaps. We also develop the elementary properties of groups to heaps.

## 2. RESEARCH METHODS

The method of this research is literature review. On [8], it has been discussed some properties of heaps and the relation between heaps and groups. Some of the properties which are related to normal sub-heaps and the quotient heaps. Therefore, this study will develop some further properties of heaps which are similar to those of groups.

## 3. RESULTS AND DISCUSSION

In this section, we discuss some further relations between groups and heaps. We will explain some brief results of the previous researches about heaps which motivate some theorems and lemma of this study. Most of the previous results are from [1], [7], [8] and [9]. The readers are considered to understand the fundamental theory of groups. Most of the concepts of groups which are used to develop the theory of heaps refer to [10], [11].

**Definition 1. [1]** A nonempty set $H$ with a ternary operation

$$[-,-,-]: H \times H \times H \longrightarrow H, \qquad (a,b,c) \mapsto [a,b,c] \tag{1}$$

is called heap if it satisfies associativity $\left([[a,b,c],d,e] = [a,b,[c,d,e]]\right)$ and Mal'cev identity $([a,b,b] = a = [b,b,a])$ for all $a,b,c,d,e \in H$. Heap $H$ together with its ternary operation are denoted by the ordered pair $(H,[-,-,-])$ or sometimes $H$ when no confusion arise. Heap $H$ is Abelian if $[a,b,c] = [c,b,a]$ for all $a,b,c \in H$.

**Lemma 2. [8]** *Let $H'$ be a heap. If $[s,t,u] = u$ or $[u,s,t] = u$ then $s = t$ for all $s,t,u \in H'$ and $[v,w,[s,t,u]] = [v,[t,s,w],u]$ for all $s,t,u,v,w \in H'$.*

The following lemmas will describe the association between heaps and groups.

**Lemma 3. [9]** *Let $(G',\cdot)$ be a group. Then $G'$ is a heap with the following ternary operation*

$$[-,-,-]: G' \times G' \times G' \longrightarrow G' \text{ where } [s,t,u] = st^{-1}u. \tag{2}$$

The following example illustrates a heap which is obtained by the given group.

**Example 1.** From the addition group $\mathbb{Z}$, we have the heap $\mathbb{Z}$ with ternary operation $[s,t,u] = s - t + u$ for all $s,t,u \in \mathbb{Z}$.

**Lemma 4. [8]** *Let $H$ be a heap and $e \in H$. Then it can be derived a group by defining the binary operation*

$$\cdot_e : H \times H \longrightarrow H \text{ where } (a,b) \mapsto [a,e,b]. \tag{3}$$

This group has an identity element $e$ and will be denoted by $(H, \cdot_e, e)$ or sometimes $H$ when no confusion can arise.

**Lemma 5. [8]** *Let $H$ be a heap and $e, f \in H$. Then $(H, \cdot_e, e)$ is isomorphic to $(H, \cdot_f, f)$ as a group.*

*The group isomorphism from $(H, \cdot_e, e)$ to $(H, \cdot_f, f)$ is defined as*

$$\tau_e^f : (H, \cdot_e, e) \longrightarrow (H, \cdot_f, f) \text{ where } s \mapsto [s,e,f]. \tag{4}$$

Our next concern will be the sub-heap.

**Definition 2. [8]** Let $(H', [-,-,-])$ be a heap and $G \subseteq H', G \neq \emptyset$. The set $G$ is a sub-heap of $H'$, denoted by $G \sqsubseteq H'$, if for every $s,t,u \in G$ then $[s,t,u] \in G$.

According to the associations of heaps and groups, we can derive some relations between sub-heaps and subgroups which are presented on the following lemma.

**Lemma 6.** *Suppose that $S$ is sub-heap of heap $H$. Then $S$ is a subgroup of $(H, \cdot_e, e)$ iff $e \in S$.*

**Proof.** If sub-heap $S$ is also a subgroup of $(H, \cdot_e, e)$, then it is obvious that $S$ has the same identity $e$. Now let sub-heap $S$ contain the element $e$. Since $S \sqsubseteq H$, then $a \cdot_e b = [a,e,b] \in S$ for all $a, b \in S$. Now let $x$ be any element of $S$. Note that the inverse of $x$ is $x^{-1} = [e,x,e]$ since

$$x \cdot_e x^{-1} = [x,e,x^{-1}] = \big[x,e,[e,x,e]\big] = e.$$

Furthermore, $[e,x,e]$ is also an element of $S$. Therefore, we can conclude that $S$ is a subgroup of $(H, \cdot_e, e)$.

**Definition 3. [8]** Assume that $H$ and $H'$ are heaps. A heap morphism from $H$ to $H'$ is a map $\psi : H \longrightarrow H'$ which preserves the ternary operation, namely

$$\psi([a,b,c]) = [\psi(a), \psi(b), \psi(c)] \tag{5}$$

for all $a, b, c \in H$.

**Lemma 7. [8]** *If $\psi$ be any heap morphism from $H$ to $H'$, then it can be defined two group homomorphims from $(H, \cdot_e, e)$ to $(H', \cdot_f, f)$ as follows*

$$\hat{\psi} : (H, \cdot_e, e) \longrightarrow (H', \cdot_f, f), \quad x \mapsto [\psi(x), \psi(e), f], \tag{6}$$

$$\hat{\psi}^\circ : (H, \cdot_e, e) \longrightarrow (H', \cdot_f, f), \quad x \mapsto [f, \psi(e), \psi(x)]. \tag{7}$$

Furthermore, if $\theta : (G, \cdot) \longrightarrow (G', \cdot)$ is a group homomorphism, then $\psi$ automatically becomes heap morphism from $(G, [-,-,-])$ to $(G', [-,-,-])$.

Now, we will define some sets which are related to heap morphisms. Let $\alpha : (H, [-,-,-]) \longrightarrow (H', [-,-,-])$ be a heap morphism and let $a$ be any element of $Im(\alpha)$. The kernel of $\alpha$ relative to $a$ is a set $\ker_a(\alpha) = \{h \in H | \alpha(h) = a\}$. The kernels of a heap morphism are normal sub-heaps. Furthermore for any $a, b \in Im(\alpha)$, $\ker_a(\alpha)$ and $\ker_b(\alpha)$ are isomorphic. Thus the kernels of a heap morphism are unique up to heap isomorphism.

The discussion of group homomorphisms and heap morphisms above motivate the following theorem.

**Theorem 1.** *Let $\phi : H \longrightarrow \tilde{H}$ be a heap morphism and $e \in H, \tilde{e} \in \tilde{H}$. Let $\hat{\phi}$ and $\hat{\phi}^\circ$ be the corresponding group homomorphisms from $(H, \cdot_e, e)$ to $(\tilde{H}, \cdot_{\tilde{e}}, \tilde{e})$. Then $ker_a(\phi) = ker(\hat{\phi}) = ker(\hat{\phi}^\circ)$ if and only if $e \in ker_a(\phi)$.*

**Proof.** Let us first prove that $e \in \ker_a(\phi)$. We know that $\ker_a(\phi)$ is a subgroup of $H$ since $\ker_a(\phi) = \ker(\hat{\phi}) = \ker(\hat{\phi}^\circ)$. Hence by **Lemma 6** it is proved that $e \in \ker_a(\phi)$. For the converse, we will prove that $\ker_a(\phi) = \ker(\hat{\phi}) = \ker(\hat{\phi}^\circ)$ whenever $e \in \ker_a(\phi)$. Let $x$ be an arbitrary element of $\ker(\hat{\phi})$ and $y$ be any element of $\ker(\hat{\phi}^\circ)$. We have the following conditions

$$\begin{aligned} \hat{\phi}(x) &= \tilde{e} \\ [\phi(x), \phi(e), \tilde{e}] &= \tilde{e} \\ \phi(x) &= \phi(e) \\ &= a \end{aligned} \tag{8}$$

and

$$\begin{aligned} \hat{\phi}^{\circ}(y) &= \tilde{e} \\ [\tilde{e}, \phi(e), \phi(y)] &= \tilde{e} \\ \phi(y) &= \phi(e) \\ &= a \end{aligned} \tag{9}$$

Those computations show us that $x, y \in \ker_a(\phi)$. Therefore, we can conclude that $\ker(\hat{\phi}) \subseteq \ker_a(\phi)$ and $\ker(\hat{\phi}^{\circ}) \subseteq \ker_a(\phi)$. Now take any element of $\ker_a(\phi)$, namely $z$. Note that $\hat{\phi}(z) = [\phi(z), \phi(e), \tilde{e}] = [a, a, \tilde{e}] = \tilde{e}$ which means $z \in \ker(\hat{\phi})$. Furthermore, note that $\hat{\phi}^{\circ}(z) = [\tilde{e}, \phi(e), \phi(z)] = [\tilde{e}, a, a] = \tilde{e}$. This implies $z \in \ker(\hat{\phi}^{\circ})$. Hence we have $\ker_a(\phi) \subseteq \ker(\hat{\phi})$ and $\ker_a(\phi) \subseteq \ker(\hat{\phi}^{\circ})$ which proving that $ker_a(\phi) = ker(\hat{\phi}) = ker(\hat{\phi}^{\circ})$.

It is known that the intersection of two subgroups is also a subgroup [12], [13]. This property also holds for sub-heaps.

**Theorem 2.** *Suppose that $S, S' \sqsubseteq H$. Then $(S \cap S') \sqsubseteq H$ iff $S \cap S' \neq \emptyset$.*

**Proof.** Since $(S \cap S') \sqsubseteq H$, by **Definition 1** $(S \cap S') \neq \emptyset$. Now we will prove the converse. It is clear that $S \cap S' \subseteq H$. Since $S \cap S' \neq \emptyset$, there exists $x, y, z \in S \cap S'$. Note that $S$ and $S'$ are sub-heaps, this implies $[x, y, z] \in S \cap S'$.

The theorem above can be generalized to the nonempty intersection of any finite collection of sub-heaps. In [14] and [15], it is asserted that group homomorphims map a subgroup to a subgroup and also assign a normal subgroup to a normal subgroup. Heap morphisms also enjoy these properties.

**Definition 4.** [8] Let $H'$ be a heap. A sub-heap $G$ is called normal if there exists $e \in G$ such that for all $x \in H'$ and $g \in G$, then there exists $g' \in G$ which satisfies $[x, e, g] = [g', e, x]$ or equivalently $\big[[x, e, g], x, e\big] = g' \in G$. We denote by $G \vartriangle H'$ the sub-heap normal $G$ of $H'$.

**Theorem 3.** *Suppose that $\phi: H \longrightarrow H'$ is a surjective heap morphism. If $N \sqsubseteq H$, then $\phi(N) \sqsubseteq H'$. In addition, if $N \vartriangle H$, then $\phi(N) \vartriangle H'$.*

**Proof.** It is clearly seen that $\phi(N)$ is a nonempty subset of $H'$. Let $\phi(a), \phi(b), \phi(c)$ be any elements of $\phi(N)$. Note that $[\phi(a), \phi(b), \phi(c)] = \phi([a, b, c]) \in \phi(N)$ since $N$ is a sub-heap of $H$. Hence $\phi(N) \sqsubseteq H'$. Now take any element $y \in H'$ and $\phi(d), \phi(g) \in \phi(N)$. The computation

$$\begin{aligned} \big[[y, \phi(d), \phi(g)], y, \phi(d)\big] &= [[\phi(x), \phi(d), \phi(g)], \phi(x), \phi(d)] \\ &= \phi(\underbrace{[[x, d, g], x, d]}_{\in N}) \end{aligned} \tag{10}$$

shows that $\phi(N) \vartriangle H'$.

It is known that on [16] and [17], the inverse image of every subgroup of group homomorphisms is subgroup. Furthermore, the inverse image of every normal subgroup of group homomorphisms is normal subgroup. The similar conditions hold for heap morphisms.

**Theorem 4.** *Let $N \sqsubseteq H'$. If $\phi: H \longrightarrow H'$ is a heap morphism, then $\phi^{-1}(N) \sqsubseteq H$. Moreover, if $N \vartriangle H'$, then $\phi^{-1}(N) \vartriangle H$.*

**Proof.** It is easily seen that $\phi^{-1}(N) \subseteq H$. If we take $x, y, z \in \phi^{-1}(N)$, we have $\phi(x), \phi(y), \phi(z) \in N$. Since $N$ is a sub-heap, then we obtain $\phi([x, y, z]) = [\phi(x), \phi(y), \phi(z)] \in N$. Therefore, $[x, y, z] \in \phi^{-1}(N)$ which means $\phi^{-1}(N)$ is a sub-heap of $H$. Furthermore, if we take any element $x$ of $H$ and $n, n'$ of $\phi^{-1}(N)$, then we get

$$\phi\big([[x, n, n'], x, n]\big) = \big[[\phi(x), \phi(n), \phi(n')], \phi(x), \phi(n)\big]. \tag{11}$$

By the normality of $N$ in $H'$, we can conclude that $\big[[\phi(x),\phi(n),\phi(n')],\phi(n),\phi(x)\big] \in N$ which implies $\big[[x,n,x'],x,n\big] \in \phi^{-1}(N)$. Hence $\phi^{-1}(N)$ is a normal sub-heap of $H$.

In this article, we will denote $N \trianglelefteq G$ whenever $N$ is a normal subgroup of $G$. The further relations between normal sub-heap and normal subgroup are stated as follows.

**Lemma 9. [8]** *Suppose that $S \neq \emptyset$ and $S \subseteq H$, where $H$ is a heap. Then these statements are equivalent.*
1. $S \vartriangle H$.
2. $S \trianglelefteq \big(H,\cdot_f,f\big)$ *for all* $f \in S$.
3. $S \trianglelefteq (H,\cdot_f,f)$, *for some* $f \in S$.

In **[18]** and **[16]**, it is explained that if we have two normal subgroups $M$ and $N$, then the intersection $M \cap N$ and the multiplication $MN$ are also normal subgroups. These properties will be developed to heap.

**Theorem 5.** *Suppose that $M,N \vartriangle H$. If $M \cap N \neq \emptyset$, then $M \cap N$ and $[M,e,N] = \{[m,e,n] | m \in M, n \in N\}$ are normal sub-heaps of $H$ for every $e \in M \cap N$.*

**Proof.** By **Theorem 2** we have $M \cap N \sqsubseteq H$. Now suppose that $a,b \in M \cap N$ and $x \in H$. Note that the normality of $M$ and $N$ in $H$ guarantee that $\big[[x,a,b],x,a\big]$ is also in $M$ and $N$. Furthermore, since $e \in M \cap N$, by **Lemma 9**, we can assume that $M$ and $N$ as normal subgroups of $H$. Now take any two elements $x = m_1 \cdot_e n_1$ and $y = m_2 \cdot_e n_2$ in $M \cdot_e N$. We have

$$\begin{aligned} x \cdot_e y^{-1} &= (m_1 \cdot_e n_1)(m_2 \cdot_e n_2)^{-1} \\ &= m_1 \cdot_e n_1 \cdot_e n_2^{-1} \cdot_e m_2^{-1} \\ &= m_1 \cdot_e \underbrace{n \cdot_e m_2^{-1}}_{\in n \cdot_e M = M \cdot_e n} \\ &= m_1 \cdot_e m \cdot_e n \in M \cdot_e N \end{aligned} \tag{12}$$

which show us that $M \cdot_e N$ is a subgroup of $(H,\cdot_e,e)$. Furthermore, the computations below

$$\begin{aligned} M \cdot_e N \cdot_e h &= M \cdot_e (N \cdot_e h) \\ &= M \cdot_e (h \cdot_e N) \\ &= (M \cdot_e h) \cdot N \\ &= (h \cdot_e M) \cdot_e N \\ &= h \cdot_e M \cdot_e N \end{aligned} \tag{13}$$

prove that $(M \cdot_e N) \trianglelefteq (H,\cdot_e,e)$. By the characterization of normal sub-heap, it is proved that $(M \cdot_e N) \vartriangle H$. The fact that $M \cdot_e N = [M,e,N]$ completes the assertion.

There are some conditions on groups theory which assert that if $K,L$ are subgroups of $H$, then the multiplication $KL$ will also be subgroup if and only if $KL = LK$. These properties are explained in **[19]**. The similar properties on heaps will be presented on the following theorem.

**Theorem 6.** *Let $K,L$ be any sub-heaps of $H$ and $e \in K \cap L$. Then $[K,e,L] = \{[k,e,l] | k \in K, l \in L\}$ is a sub-heap of $H$ if and only if $[K,e,L] = [L,e,K]$.*

**Proof.** Assume that $[K,e,L]$ is a sub-heap of $H$. Since $e \in K \cap L$, we have $k = [k,e,e] \in [K,e,L]$, $l = [e,e,l] \in [K,e,L]$, and $e \in [K,e,L]$. It follows that $[l,e,k] \in [K,e,L]$. Thus $[L,e,K] \subseteq [K,e,L]$.
Now let $x$ be any element of $[K,e,L]$. We can consider $[K,e,L] = K \cdot_e L$ as a subgroup of $(H,\cdot_e,e)$. Therefore, $[e,x,e] = x^{-1} \in [K,e,L]$. We can write $[e,x,e] = [k,e,l]$ for some $k \in K$ and $l \in L$. Note that

$$\begin{aligned} x &= [e,e,x] \\ &= \big[e,e,[x,e,e]\big] \\ &= \big[e,[e,x,e],e\big] \\ &= \big[e,[k,e,l],e\big] \\ &= \big[e,l,[e,k,e]\big] \\ &= \Big[e,l,\big[e,e,[e,k,e]\big]\Big] \\ &= \Big[\underbrace{[e,l,e]}_{\in K},e,\underbrace{[e,k,e]}_{\in L}\Big] \in [L,e,K]. \end{aligned} \tag{14}$$

We have thus proved that $[K,e,L] \subseteq [L,e,K]$.

Conversely, let $[K, e, L] = [L, e, K]$. We begin by proving $[K, e, L]$ is a subgroup of $(H, \cdot_e, e)$. Let $a, b$ be any elements of $[K, e, L]$. We can write $a = [k_1, e, l_1] = k_1 \cdot_e l_1$ and $b = [k_2, e, l_2] = k_2 \cdot_e l_2$ for some $k_1, k_2 \in K$ and $l_1, l_2 \in L$. Note that

$$\begin{aligned} a \cdot_e b^{-1} &= (k_1 \cdot_e l_1) \cdot_e (k_2 \cdot_e l_2) \\ &= k_1 \cdot_e (l_1 \cdot k_2) \cdot_e l_2 \\ &= k_1 \cdot_e (k_3 \cdot_e l_3) \cdot_e l_2 \text{ for some } k_3 \in K, l_3 \in L \\ &= \underbrace{(k_1 \cdot_e k_3)}_{\in K} \cdot_e l_3 \in K \cdot_e L = [K, e, L]. \end{aligned} \tag{15}$$

Thus $[K, e, L]$ is a subgroup of $(H, \cdot_e, e)$. Next we will prove that $[K, e, L]$ is a sub-heap of $H$. To do this, take $a, b, c \in [K, e, L]$. Note that $[K, e, L]$ being subgroup implies

$$\begin{aligned} [a, b, c] &= \big[[a, b, e], e, c\big] \\ &= \big[a, e, [e, b, e]\big] \cdot_e c \\ &= a \cdot_e [e, b, e] \cdot_e c \\ &= a \cdot_e b^{-1} \cdot_e c \in [K, e, L]. \end{aligned} \tag{16}$$

Hence, $[K, e, L]$ is a sub-heap of $H$.

## 4. CONCLUSIONS

There are some elementary properties of heaps which are obtained from those of groups, such as the property associated to sub-heaps, normal sub-heaps, and heap morphism. We give some characterizations between sub-heap and subgroup, kernel of heap morphisms and the corresponding group homomorphisms. Furthermore, a surjective heap morphism preserves the normality of sub-heap.

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