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EDGE IRREGULAR REFLEXIVE LABELING ON MONGOLIAN TENT GRAPH ($M_{m,3}$) AND DOUBLE QUADRILATERAL SNAKE GRAPH

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ABSTRACT

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Let G be an undirected, connected, and simple graph with edges set $E(G)$and vertex set $V(G)$. An edge irregular reflexive k-labeling f is one in which the label for each edge is an integer number $\{1,2,\dots,k_e\}$ and the label for each vertex is an even integer number $\{0,2,4,\dots,2k_v\}$, $k = max\{k_e, 2k_v\}$. This type of labeling results in distinct weights for each edge. The weight of an edge xy in a graph G with labeling f, indicated by $wt\ (xy)$, is the total of the labels on the vertex that are incident to the edge as well as the edge label. The minimum value k of the largest label in the graph G is referred to as $res\ (G)$, which stands for the reflexive edge strength of the graph G. The topic of edge irregular reflexive k-labeling for Mongolian tent graph $(M_{m,n})$ and double quadrilateral snake graph $(D(Q_n))$ will be discussed in this paper. The res $(M_{m,n})$,$m \geq 2$,$n = 3$ has been obtained that is $\left\lceil\frac{5m-1}{3}\right\rceil$ for $5m-1 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil\frac{5m-1}{3}\right\rceil + 1$ for $5m-1 \equiv 2,3\ (mod\ 6)$. Also the res $(D(Q_n))$,$n \geq 2$ has been obtained that is $\left\lceil\frac{7n-7}{3}\right\rceil$ for $7n-7 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil\frac{5m-1}{3}\right\rceil + 1$ for $7n-7 \equiv 2,3\ (mod\ 6)$.

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1. INTRODUCTION

The graphs discussed in this article are undirected, connected, and simple graphs. Labeling is one of the subjects in graph theory. A labeling of a graph is a mapping from graph elements into positive or non-negative integers [1]. There are many methods for labeling graphs, which are broken down into vertex, edge, and edge and vertex labeling, also known as complete labeling. Graph labeling has evolved to include a number of different varieties, one of which is the irregular total k-labeling [2].

Bača *et al.* [3] defined total irregular k-labeling divided into edge irregular total k-labeling and vertex irregular total k-labeling. Then in 2017, Ryan *et al.* [4] introduced a new concept of irregular total k-labeling, that is vertex irregular reflexive k-labeling and edge irregular reflexive k-labeling on a graph. An edge irregular reflexive k-labeling on a graph G is a labeling that takes positive integers from 1 to k_e as edge labels and positive even integers from 0 to $2k_v$ as vertex labels so that the weight on each edge of the graph is different.

Ryan *et al.* [4] define a vertex labeling $f_v : V(G) \to \{0,2,\dots,2k_v\}$ and edge labeling $f_e : E(G) \to \{1,2,\dots,k_e\}$ with $k = max\{k_e, 2k_v\}$. The weight of edge xy in G with labeling f, denoted by $wt(xy)$ is defined as $wt(xy) = f_v(x) + f_e(xy) + f_v(y)$. The minimum value k of the largest label of vertex and edge is referred to as $res(G)$, which stands for the reflexive edge strength of the graph G [5]. The following lemma is given according to Ryan *et al.* in Bača *et al.* [4].

Lemma 1. *For every graph* G,

$$res(G) \geq \begin{cases} \left\lceil \dfrac{|E(G)|}{3} \right\rceil, & \text{for } |E(G)| \not\equiv 2,3 \ (mod\ 6), \\ \left\lceil \dfrac{|E(G)|}{3} \right\rceil + 1, & \text{for } |E(G)| \equiv 2,3 \ (mod\ 6). \end{cases}$$

The lower bound for $res(G)$ resulted from the facts that the smallest edge weight under an edge irregular reflexive labeling is one and the basis of the maximal edge weights, so $|E(G)|$ can only be obtained as the sum of three values, at least two of which must be even.

Several result in $res(G)$ have been obtained, banana tree graph $B_{2,n}$ and $B_{3,n}$ [6], categorical product of two paths [7], disjoint union of generalized petersen graph [8], palm tree graphs $C_3 - B_{2,r}$ and $C_3 - B_{3,r}$ [9], caterpillar graphs [10] and then in 2020, Agustin *et al.* [11] determined the $res(G)$ of broom graphs, generalized sub-divided star graph, and double star graph, in the same year Indriati *et al.* [12] determined corona of path and complete graph K_1 and corona of path and path graph P_2. Then in 2021, Setiawan and Indriati [13] determined corona of cycle and null graph. Because mongolian tent graph $M_{m,3}$ with $m \geq 2$ [14] and double quadrilateral snake graph $D(Q_n)$ with $n \geq 2$ [15] it has been studied in other labeling while in this labeling has not, so in this paper we study these graphs for edge irregular reflexive labeling and determine these reflexive edge strength.

2. RESEARCH METHODS

A literature review employing references from journals, books, or publications on edge irregular reflexive k-labeling was the research methodology used in this study. From this method, it can be determined $res(G)$ of mongolian tent graph $M_{m,3}$ and double quadrilateral snake graph $D(Q_n)$. The methods used in this research are

1. Determine the lower bounds of $res(G)$ of mongolian tent graph $M_{m,3}$ and double quadrilateral snake graph $D(Q_n)$ based on Lemma 1 is given according to Ryan *et al.* in Bača *et al.* [4]
2. Labeling the mongolian tent $M_{m,3}$ and double quadrilateral snake graph $D(Q_n)$ that satisfies the predefined lower bound.
3. Calculating the weight of each edge of mongolian tent graph $M_{m,3}$ and double quadrilateral snake graph $D(Q_n)$ so that the weight of each edge is different.
4. Find the general pattern res of mongolian tent graph $M_{m,3}$and double quadrilateral snake graph $D(Q_n)$.

3. RESULTS AND DISCUSSION

In this chapter, results and discussion are given about reflexive edge strength on mongolian tent graph $M_{m,3}$ and double quadrilateral snake graph $D(Q_n)$.

3.1 Mongolian Tent Graph $M_{m,3}$

The process of creating a mongolian tent graph involves adding one additional vertex above the grid and connecting each odd vertex in the first row of $P_m \times P_n$ to the new vertex [15]. Mongolian tent graph is denoted by $M_{m,n}$where $m \geq 2$ and $n \geq 3$.

A mongolian tent graph $M_{m,3}$ has $3m+1$ order and $5m-1$ size. Let the vertex set of $M_{m,3}$ be $V(M_{m,3}) = \{u\} \cup \{v_{i,1}, v_{i,2}, v_{i,3} : 1 \leq i \leq m\}$ and the edge set of $M_{m,3}$ be $E(M_{m,3}) = \{v_{i,1}v_{i,2}, v_{i,2}v_{i,3} : 1 \leq i \leq m\} \cup \{v_{i,1}v_{i+1,1}, v_{i,2}v_{i+1,2}, v_{i,3}v_{i+1,3} : 1 \leq i \leq m-1\} \cup \{uv_{1,j} : j\ odd\}$. The reflexive edge strength of mongolian tent graph $M_{m,3}$ can be obtained through **Theorem 1**.

Theorem 1. *For all positive integers $m \geq 2$ and $n = 3$,*

$$res(M_{m,3}) = \begin{cases} \left\lceil \frac{5m-1}{3} \right\rceil, \text{if } 5m-1 \not\equiv 2,3\ (mod\ 6), \\ \left\lceil \frac{5m-1}{3} \right\rceil + 1, \text{if } 5m-1 \equiv 2,3\ (mod\ 6). \end{cases}$$

Proof. Because mongolian tent graph $M_{m,3}$has $5m-1$ edges, then by [4,Lemma 1] we get

$$res(M_{m,3}) \geq \begin{cases} \left\lceil \frac{5m-1}{3} \right\rceil, \text{if } 5m-1 \not\equiv 2,3\ (mod\ 6), \\ \left\lceil \frac{5m-1}{3} \right\rceil + 1, \text{if } 5m-1 \equiv 2,3\ (mod\ 6). \end{cases}$$

The lower bound in **Theorem 1** is the same as the lower bound of $res(M_{m,3})$ shown by Ryan *et al.* in Bača *et al.* [4].

Next, we have shown an upper bound on the labeling of edge irregular reflexive labelling in mongolian tent graph $M_{m,3}$. Let us construct k-labeling f with $k = \left\lceil \frac{5m-1}{3} \right\rceil$ for $5m-1 \not\equiv 2,3\ (mod\ 6)$ and $k = \left\lceil \frac{5m-1}{3} \right\rceil + 1$ for $5m-1 \equiv 2,3\ (mod\ 6)$,

$$f(u) = \begin{cases} \left\lceil \frac{2m}{3} \right\rceil + m, m \equiv 1,2\ (mod\ 6), \\ 2m - \left\lceil \frac{m}{3} \right\rceil, m \equiv 0,4,5\ (mod\ 6), \\ \left\lceil \frac{2m}{3} \right\rceil + m + 1, m \equiv 3\ (mod\ 6). \end{cases}$$

$$f(v_{i,1}) = f(v_{i,2}) = f(v_{i,3}) = \begin{cases} 2m - 2i, \left\lceil \frac{m-3}{6} \right\rceil + 1 \leq i \leq m, m > 3, \text{or } 1 \leq i \leq m, m = 2,3, \\ 2m - \left\lceil \frac{m+1}{3} \right\rceil, m \equiv 1\ (mod\ 6), m > 3, 1 \leq\ i \leq \left\lceil \frac{m-3}{6} \right\rceil, \\ 2m - \left\lceil \frac{m-1}{3} \right\rceil, m \equiv 2,3\ (mod\ 6), m > 3, 1 \leq\ i \leq \left\lceil \frac{m-3}{6} \right\rceil, \\ 2m - \left\lceil \frac{m}{3} \right\rceil, m \equiv 0,4,5\ (mod\ 6), m > 3, 1 \leq\ i \leq \left\lceil \frac{m-3}{6} \right\rceil. \end{cases}$$

$$f(v_{i,1}v_{i,2}) = \begin{cases} m-i+1, 1 \le i \le m, m \le 9 \text{ or } \left\lfloor \frac{m-4}{6} \right\rfloor + 1 \le i \le m, m > 9, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor, m \equiv 1 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor - 1, m \equiv 2{,}3 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor + 2, m \equiv 4 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor + 1, m \equiv 0{,}5 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor. \end{cases}$$

$$f(v_{i,2}v_{i,3}) = \begin{cases} m-i+2, 1 \le i \le m, m \le 9 \text{ or } \left\lfloor \frac{m-4}{6} \right\rfloor + 1 \le i \le m, m > 9, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor + 1, m \equiv 1 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor, m \equiv 2{,}3 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor + 3, m \equiv 4 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lfloor \frac{m}{3} \right\rfloor + 2, m \equiv 0{,}5 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor. \end{cases}$$

$$f(v_{i,1}v_{i+1,1}) = \begin{cases} m-i, 1 \le i \le m-1, m \le 9 \text{ or } \left\lfloor \frac{m-4}{6} \right\rfloor + 1 \le i \le m-1, m > 9, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil - 2, m \equiv 0 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil - (t+1), m \equiv t \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, t = 1{,}2{,}3, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil, m \equiv 4 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil - 1, m \equiv 5 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor. \end{cases}$$

$$f(v_{i,2}v_{i+1,2}) = \begin{cases} m-i+1, 1 \le i \le m-1, m \le 9 \text{ or } \left\lfloor \frac{m-4}{6} \right\rfloor + 1 \le i \le m-1, m > 9, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil - 1, m \equiv 0 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil - r, m \equiv r \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, r = 1{,}2{,}3, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil + 1, m \equiv 4 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil, m \equiv 5 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor. \end{cases}$$

$$f(v_{i,3}v_{i+1,3}) = \begin{cases} m-i+2, 1 \le i \le m-1, m \le 9 \text{ or } \left\lfloor \frac{m-4}{6} \right\rfloor + 1 \le i \le m-1, m > 9, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil, m \equiv 0 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil - (s-1), m \equiv s \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, s = 1{,}2{,}3, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil + 2, m \equiv 4 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor, \\ 2m-5i-\left\lceil \frac{m}{3} \right\rceil + 1, m \equiv 5 \ (mod\ 6), m > 9, 1 \le\ i \le \left\lfloor \frac{m-4}{6} \right\rfloor. \end{cases}$$

For j odd,

$$f(uv_{1,j}) = \begin{cases} m + \left\lfloor \frac{j}{3} \right\rfloor, m \leq 3, \\ 2m - \left\lceil \frac{m+2t+4}{3} \right\rceil + \left\lfloor \frac{j}{3} \right\rfloor, m \equiv t \ (mod\ 6), t = 0,1,2,3, m > 3, \\ 2m - \left\lceil \frac{m}{3} \right\rceil + \left\lfloor \frac{j}{3} \right\rfloor, m \equiv 4 \ (mod\ 6), m > 3, \\ 2m - \left\lceil \frac{m+2}{3} \right\rceil + \left\lfloor \frac{j}{3} \right\rfloor, m \equiv 5 \ (mod\ 6), m > 3. \end{cases}$$

Based on the proof of the lower and upper bound of $res\ (M_{m,3})$ for $m \geq 2$, obtained the maximum label of the vertex label which is an even integer and edge labels that are positive integers, namely $\left\lceil \frac{5m-1}{3} \right\rceil$ for $5m-1 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil \frac{5m-1}{3} \right\rceil + 1$ for $5m-1 \equiv 2,3\ (mod\ 6)$. Furthermore, it is shown edge weight for mongolian tent graph $M_{m,3}$ as follows,

$$w_t(v_{i,1}v_{i,2}) = 5m - 5i + 1, \text{for } m \geq 2, 1 \leq i \leq m.$$
$$w_t(v_{i,2}v_{i,3}) = 5m - 5i + 2, \text{for } m \geq 2, 1 \leq i \leq m.$$
$$w_t(v_{i,1}v_{i+1,1}) = 5m - 5i - 2, \text{for } m \geq 2, 1 \leq i \leq m-1.$$
$$w_t(v_{i,2}v_{i+1,2}) = 5m - 5i - 1, \text{for } m \geq 2, 1 \leq i \leq m-1.$$
$$w_t(v_{i,3}v_{i+1,3}) = 5m - 5i, \text{for } m \geq 2, 1 \leq i \leq m, -1.$$
$$w_t(uv_{1,j}) = \begin{cases} 5m - 2, \text{for } j = 1, \\ 5m - 1, \text{for } j = 3. \end{cases}$$

Based on the research that has been done, it can be concluded that each edge has a different weight, the lower bound and upper bound are equal to $res(M_{m,3})$ so that f satisfies the edge irregular reflexive k-labeling element and has a reflexive edge strength (res) according to **Theorem 1**. Thus, the theorem is proven.

Example 1. The edge irregular reflexive 7-labeling of mongolian tent graph $M_{4,3}$. **Figure 1** shows **Example 1**, the red color is edge weights, blue color is edge labels and black color is vertex labels, while the black letters are the names of the vertex and edge labels of mongolian tent graph $M_{4,3}$.

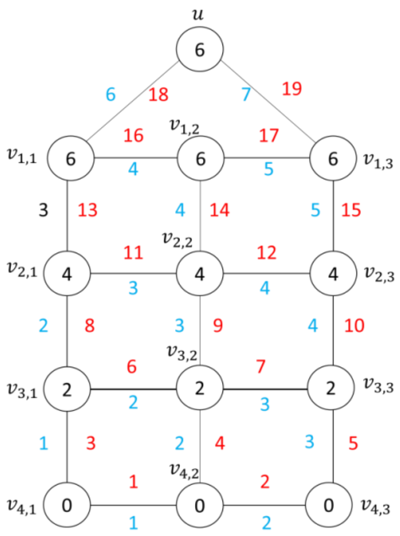

Figure 1. Edge irregular reflexive 7-labeling of mongolian tent $M_{4,3}$ graph

3.2 Double Quadrilateral Snake Graph $D(Q_n)$

A double quadrilateral snake graph is is obtained form a path $u_1, u_2, \dots, u_n$ by joining each of vertex u_i and u_{i+1}, $i = 1,2,\dots,n-1$ to new vertex v_i, x_i and to the new vertex w_i and y_i respectively and adding an edge between each pair of vertex (v_i, w_i) and (x_i, y_i) [17]. A double quadrilateral snake graph is denoted by $D(Q_n)$ for $n \geq 2$.

A double quadrilateral snake graph has $5n-4$ order and $7n-7$ size. Let the vertex set of $D(Q_n)$ be $V(D(Q_n)) = \{u_i: 1 \le i \le n\} \cup \{v_i, x_i, w_i, y_i : 1 \le i \le n-1\}$ and the edge set of $D(Q_n)$ be $E(D(Q_n)) = \{u_i x_i, u_i u_{i+1}, u_{i+1} w_i, u_i v_i, x_i y_i, u_{i+1} y_i: 1 \le i \le n-1\}$. The reflexive edge strength of double quadrilateral snake graph $D(Q_n)$ can be obtained through **Theorem 2**.

Theorem 2. *For all positive integers* $n \ge 2, res(D(Q_n)) = \begin{cases} \left\lceil \frac{7n-7}{3} \right\rceil, \text{if } 7n-7 \not\equiv 2,3 \ (mod\ 6), \\ \left\lceil \frac{7n-7}{3} \right\rceil + 1, \text{if } 7n-7 \equiv 2,3 \ (mod\ 6). \end{cases}$

Proof. Because double quadrilateral snake graph $D(Q_n)$ has $7n-7$ edges, then by **[4, Lemma 1]** we get

$$res(D(Q_n)) \ge \begin{cases} \left\lceil \frac{7n-7}{3} \right\rceil, \text{if } 7n-7 \not\equiv 2,3 \ (mod\ 6), \\ \left\lceil \frac{7n-7}{3} \right\rceil + 1, \text{if } 7n-7 \equiv 2,3 \ (mod\ 6). \end{cases}$$

Next, the upper bound on $res(D(Q_n))$ is proved. To prove the upper bound, let us construct k-labeling f with $k = \left\lceil \frac{7n-7}{3} \right\rceil$ for $7n-7 \not\equiv 2,3 \ (mod\ 6)$ and $k = \left\lceil \frac{7n-7}{3} \right\rceil + 1$ for $7n-7 \equiv 2,3 \ (mod\ 6)$, for $1 \le i \le n$,

$$f(u_i) = \begin{cases} 0, i = 1, \\ \left\lfloor \frac{4i-3}{3} \right\rfloor + i, i \equiv 0,2 \ (mod\ 3), i \text{ odd or } i \equiv 0 \ (mod\ 3), i \text{ even}, \\ \left\lfloor \frac{4i-4}{3} \right\rfloor + i, i \equiv 0 \ (mod\ 3), i \text{ even}, \\ \left\lfloor \frac{4i-3}{3} \right\rfloor + i - 1, i \equiv 1 \ (mod\ 3), i \text{ odd or } i \equiv 2 \ (mod\ 3), i \text{ even}. \end{cases}$$

For $1 \le i \le n-1$,

$$f(v_i) = f(x_i) = \begin{cases} 0, i = 1, \\ \left\lfloor \frac{4i-3}{3} \right\rfloor + i, i \equiv 0,2 \ (mod\ 3), i \text{ odd or } i \equiv 1 \ (mod\ 3), i \text{ even}, \\ \left\lfloor \frac{4i-4}{3} \right\rfloor + i, i \equiv 0 \ (mod\ 3), i \text{ even}, \\ \left\lceil \frac{4i-3}{3} \right\rceil + i, i \equiv 2 \ (mod\ 3), i \text{ even}, \\ \left\lceil \frac{4i-4}{3} \right\rceil + i - 1, i \equiv 1 \ (mod\ 3), i \text{ odd}. \end{cases}$$

$$f(w_i) = f(y_i) = \begin{cases} \left\lceil \frac{4i-1}{3} \right\rceil + i + 1, i \equiv 0 \ (mod\ 3), i \text{ odd or } i \equiv 1 \ (mod\ 3), i \text{ even}, \\ \left\lceil \frac{4i-1}{3} \right\rceil + i, i \equiv 0 \ (mod\ 3), i \text{ even or } i \equiv 1,2 (mod\ 3), i \text{ odd}, \\ \left\lceil \frac{4i-3}{3} \right\rceil + i - 1, i \equiv 2 \ (mod\ 3), i \text{ even}. \end{cases}$$

$$f(u_i x_i) = f(u_i u_{i+1}) = (u_{i+1} w_i) = \begin{cases} \left\lfloor \frac{4i-7}{3} \right\rfloor + i, i \equiv 0 \ (mod\ 3), i \text{ odd or} \\ \quad i \equiv 1 \ (mod\ 3), i \text{ even}, \\ \left\lceil \frac{4i-5}{3} \right\rceil + i, i \equiv 0 \ (mod\ 3), i \text{ even or} \\ \quad i \equiv 2 (mod\ 3), \\ \left\lceil \frac{4i-3}{3} \right\rceil + i, i \equiv 1 \ (mod\ 3), i \text{ odd}. \end{cases}$$

$$f(u_iv_i) = f(v_iw_i) = \begin{cases} \left\lfloor \frac{4i-11}{3} \right\rfloor + i, i \equiv 0 \ (mod\ 3), i \text{ odd}, \\ \left\lceil \frac{4i-7}{3} \right\rceil + i, i \equiv 0 \ (mod\ 3), i \text{ even}, \\ \left\lfloor \frac{4i-3}{3} \right\rfloor + i, i \equiv 1 \ (mod\ 3), i \text{ odd}, \\ \left\lceil \frac{4i-11}{3} \right\rceil + i, i \equiv 1 \ (mod\ 3), i \text{ even}, \\ \left\lfloor \frac{4i-7}{3} \right\rfloor + i, i \equiv 2 \ (mod\ 3). \end{cases}$$

$$f(x_iy_i) = f(y_iu_{i+1}) = \begin{cases} \left\lfloor \frac{4i-5}{3} \right\rfloor + i, i \equiv 0 \ (mod\ 3), i \text{ odd}, \\ \left\lceil \frac{4i-1}{3} \right\rceil + i, i \equiv 0 \ (mod\ 3), i \text{ even}, \\ \left\lceil \frac{4i}{3} \right\rceil + i, i \equiv 1 \ (mod\ 3), i \text{ odd}, \\ \left\lceil \frac{4i-5}{3} \right\rceil + i, i \equiv 1 \ (mod\ 3), i \text{ even}, \\ \left\lfloor \frac{4i-1}{3} \right\rfloor + i, i \equiv 2 \ (mod\ 3). \end{cases}$$

Based on the proof of the lower and upper bound of $res\ (D(Q_n))$ for $n \geq 2$, obtained the maximum label of the vertex label which is an even integer and edge labels that are positive integers, namely $\left\lceil \frac{7n-7}{3} \right\rceil$ for $7n - 7 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil \frac{7n-7}{3} \right\rceil + 1$ for $7n - 7 \equiv 2,3\ (mod\ 6)$. Furthermore, it is shown edge weight for double quadrilateral snake graph $D(Q_n)$ as follows,

$$w_t(u_iu_{i+1}) = 7i - 3, \text{for } n \geq 2, 1 \leq i \leq n-1.$$
$$w_t(u_iv_i) = 7i - 6, \text{for } n \geq 2, 1 \leq i \leq n-1.$$
$$w_t(v_iw_i) = 7i - 4, \text{for } n \geq 2, 1 \leq i \leq n-1.$$
$$w_t(u_ix_i) = 7i - 5, \text{for } n \geq 2, 1 \leq i \leq n-1.$$
$$w_t(x_iy_i) = 7i - 2, \text{for } n \geq 2, 1 \leq i \leq n-1.$$
$$w_t(u_{i+1}w_i) = 7i - 1, \text{for } n \geq 2, 1 \leq i \leq n-1.$$
$$w_t(u_{i+1}y_i) = 7i, \text{for } n \geq 2, 1 \leq i \leq n-1.$$

Based on the research that has been done, it can be concluded that each edge has a different weight, the lower bound and upper bound are equal to $res(D(Q_n))$ so that f satisfies the edge irregular reflexive k-labeling element and has a reflexive edge strength (res) according to **Theorem 2**. Thus the theorem is proven.

Example 2. The edge irregular reflexive 8-labeling of double quadrilateral snake graph $D(Q_4)$. **Figure 2** shows **Example 2**, the red color is edge weights, blue color is edge labels and black color is vertex labels, while the black letters are the names of the vertex and edge labels of double quadrilateral snake graph $D(Q_4)$.

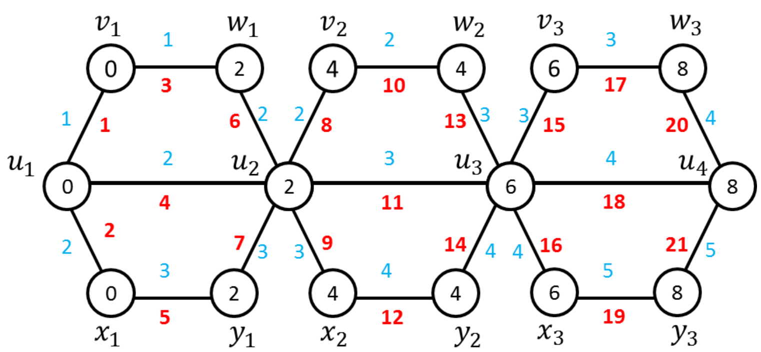

Figure 2. Edge Irregular Reflexive 8-Labeling Of Double Quadrilateral Snake Graph $D(Q_4)$.

4. CONCLUSIONS

Based on this discussion, the reflexive edge strength of mongolian tent graph $M_{m,3}$ is $\left\lceil\frac{5m-1}{3}\right\rceil$ for $5m-1 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil\frac{5m-1}{3}\right\rceil+1$ for $5m-1 \equiv 2,3\ (mod\ 6)$. While the reflexive edge strength of double quadrilateral snake graph $D(Q_n)$ is $\left\lceil\frac{7n-7}{3}\right\rceil$ for $7n-7 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil\frac{5m-1}{3}\right\rceil+1$ for $7n-7 \equiv 2,3\ (mod\ 6)$. The author hopes that those interested in edge irregular reflexive k-labeling and reflexive edge strength on graphs will conduct research to determine the reflexive edge strength of mongolian tent $M_{m,n}$ graphs for $m \geq 2, n \geq 3$.

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