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WEIBULL-POISSON DISTRIBUTION AND THEIR APPLICATION TO SYSTEMATIC PARALLEL RISK

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ABSTRACT

The Weibull-Poisson distribution represents a continuous distribution type applicable to various forms of hazard, including monotone up, monotone down, and upside-down bathtub shapes that ascend. The distribution characterizes lifetimes and can effectively model failures within a series of systems, which evolves from the Exponential-Poisson distribution. This distribution emerges through the compounding of the Weibull Distribution and Zero Truncated Poisson Distribution. The compounding itself integrates several mathematical properties, such as statistical order and Taylor's number expansion, to reach its final form. Alongside the formulation of the Weibull-Poisson distribution, this paper includes the probability density function, distribution function, r-th moment, r-th central moment, mean, and variance. For illustration, the Weibull-Poisson distribution is applied to guinea pig survival data after being infected with Turblece virus Bacilli.

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1. INTRODUCTION

Lifetime data is data consisting of the time until an event or event occurs against an object/individual in the population, for example, the recovery time for COVID-19 patients in an age group or the time until a machine fails to operate. Lifetime data analysis has an important role in the world of science, especially in engineering and medical fields.

Analysis of reliability is an analysis of the possibility of failure of a system. There are three types of resistance systems, namely series systems, parallel systems, and series-parallel systems. In a series system, if one component fails, it will cause failure of the entire system. In a parallel system, the system can fail if and only if all components fail, while a series-parallel system is a combination of both parallel and series systems. Compounding distributions can be built to make either parallel or series systems.

The lifetime data of a series system can be modeled by modifying the existing lifetime distribution. There are various ways to form a new distribution based on pre-existing distributions, one of which is the compound method. A compound distribution in statistics refers to a probability distribution that arises from the combination of two or more probability distributions. It represents a situation where the random variable of interest is affected by multiple underlying processes or sources of variability. First, there was an introduction of geometric exponential 1distributions [1] by compounding exponential distributions and geometric distributions. The geometric exponential distribution has a decreasing failure rate. Following the idea, the Exponential-Poisson distribution [2] was introduced by performing a compounding technique between the Exponential distribution and the ZT-Poisson distribution. Then, the Exponential-Logarithmic distribution was introduced by compounding the exponential distribution and the logarithmic distribution [3]. All the compounding distributions that have been mentioned are based on the Exponential distribution, and they have a monotonous failure rate.

The Weibull distribution is a generalized form of the Exponential distribution. This distribution is considered superior to the exponential because of the hazard shape by the limited Exponential distribution. Based on this, many previous researchers replaced the Exponential distribution in the compounding method with the Weibull distribution. This was done by researcher Baretto Souza who introduced the Weibull-Geometric distribution [4] as a generalized form of the Exponential-Poisson distribution by Adamidis. Then, Ciumara and Preda introduced the Weibull-Logarithmic distribution [5] as a more flexible form of the Exponential-Logarithmic distribution. Furthermore, Wanbo Lu and Daimin Shi introduced the Weibull-Poisson distribution [6] as a form that is more flexible than the Exponential-Poisson distribution.

From the research mentioned above, the compound method of a certain distribution with the Weibull distribution produces a more flexible form compared to the compound method with an exponential distribution. This can be seen from the results of the model fit with lifetime data and variations in the shape of the hazard function from both distributions.

The Weibull-Poisson distribution introduced by Lu and Shi [6] has a flexible form of the hazard function, namely increasing monotone, decreasing monotone and unimodal form. An example of an application of the Weibull-Poisson distribution is analyzing data on the time to failure of an aircraft engine after it has been repaired.

In this paper, we will discuss the Weibull and ZT-Poisson distributions. We will construct the Weibull-Poisson distribution using the compounding method introduced by Baretto-Souza. Next, we will discuss the characteristics of the Weibull-Poisson distribution, which include the probability density function, cumulative distribution function, survival function, hazard function, r-th moment, mean, and variance. In addition, we will discuss the estimation of the parameters of the Weibull-Poisson distribution using the maximum likelihood method. Finally, the Weibull-Poisson distribution will be illustrated using real data.

2. RESEARCH METHODS

2.1 Weibull Distribution

The Weibull distribution is a continuous distribution with two scale parameters, namely α and β. The random variable X, denoted as $X \sim$ Weibull (α, β), has the probability density function as:

$$f(x) = \alpha\beta x^{\alpha-1}\, e^{\exp(-\beta x^{\alpha})} \tag{1}$$

The Weibull distribution has various forms of hazard functions, namely monotonically increasing, monotonically decreasing, and constant.

2.2 ZT-Poisson Distribution

The Zero-Truncated Poisson distribution (ZT-Poisson) is a Poisson distribution that models the number of occurrences of an event in a certain time interval or space with the assumption that the event has occurred at least once so that a zero value is impossible and only considers positive values. The random variable $\boldsymbol{g(x;\mu)}$ has a ZT-Poisson($\boldsymbol{\mu}$) distribution if it has the probability density function as:

$$g(x;\mu) = \frac{\mu^x exp\,(-\mu)}{x!(1-exp\,(-\mu))},\ \text{where } x = 1,2,3,\ldots \tag{2}$$

Probability density function stated at **Equation (2)** can be written in Gamma function as:

$$g(x;\mu) = \frac{\mu^x e^{-\mu}}{\Gamma(x+1)(1-e^{-\mu})},\ \text{where } x = 1,2,\ldots \tag{3}$$

2.3 Cumulative Distribution Function and Probability Density Function of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be identical and independent continuous random variables with the probability density function $f_X(x)$ and the cumulative distribution function $F_X(x)$. Suppose Y_1 is the smallest random variable of all X_i, Y_2 is the 2nd smallest random variable of all X_i, and Y_n is the largest random variable of all X_i. Thus, $Y_1 < Y_2 < \ldots < Y_n$ denotes the random variables $X_1, X_2, \ldots, X_n$ which are ordered from the smallest to largest. In another form Y_1 can be written as $Y_1 = Min(X_1, X_2, \ldots, X_n)$.

Where the cumulative distribution function of the random variable Y_1 , as theory from [9] can be written as follows:

$$F_{Y_1}(y) = 1 - [1 - F_Y(y)]^n \tag{4}$$

For example, $f_{Y_1}(y)$ are probability density function for Y_1, hence it can be derived from differentiation of **Equation (4)** can be written as follow:

$$g_{Y_1}(y) = n[1 - F_Y(y)]^{n-1} f_Y(y) \tag{5}$$

2.4 Taylor Binomial

Taylor Binomial will be used in constructing the Weibull-Poisson Distribution. The Taylor series is an infinite series for representing a mathematical function into an infinite number of countable terms. The Taylor series has the general form as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots \tag{6}$$

For example, $f(x) = e^x$, with the Taylor series define in **Equation (6)** we can derive $f(x)$ as:

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{\Gamma(n)} \tag{7}$$

2.5 Newton-Raphson's Numerical Method

In completing this paper, the equations generated by the maximum likelihood method cannot be analyzed analytically because they have a complex and nonlinear shape. Therefore, we use New Raphson's method, a derivative of Taylor's Theorem and has the form of the recursive formula below to find a solution $f(x) = 0$ [10].

$$x_n = x_n - 1 - \frac{f(x_n - 1)}{f'(x_n - 1)}, n > 1 \tag{8}$$

2.6 Data Sources, Collection and Data Test Method

Data Sources and Collection

The data is retrieved from the journal by Bjerkedal [6]. In these data, we obtained the life span in days of 72 guinea pigs after being infected by a virus.

Data Test Method: Kolmogorov-Smirnov Method

The Kolmogorov-Smirnov model fit test is a test to match the data to an alleged distribution. [11] This test uses the distance between the empirical cumulative distribution function and the cumulative distribution function of the alleged distribution.

For example, $X_1, X_2, \ldots, X_n$ is a random variable that has a population size of n which has a distribution function $F(x)$. This distribution function will be estimated by an empirical distribution F_n which is defined as:

$$F_n \leq \frac{\text{amount of } x_i \leq x}{n} \tag{9}$$

It will be tested whether the data with the empirical distribution function F_n comes from the cumulative distribution function $F(x)$ or not. This is stated by the following hypothesis:

H_0 : data comes from a population that has a cumulative distribution function $F(x)$

H_1 : data does not come from a population that has a cumulative distribution function $F(x)$

The above hypothesis is tested using the following test statistics:

$$T = \sup_x |F_n(x) - F^*(x)| \tag{10}$$

where $F*(x)$ is the cumulative distribution function obtained by parameter estimation. H_0 will be rejected if the T-value is greater than the critical value.

3. RESULTS AND DISCUSSION

3.1 Writing Mathematical Equations

In a series component system, the system will work if and only if all the components work. Therefore, the time to failure of a series system is the same as the time to failure of the first component, provided that the component that fails the first time occurs randomly.

3.2 Deriving Probability Density Function Compounding Method with ZT-Poisson

The compounding method was first introduced by Adamidis and Loukas [1] to form the Exponential-Geometric distribution. This method was introduced to model the lifetime of a system with Z components where Z is a discrete random variable and Y_i for $i = 1,2,..,Z$ is a continuous random variable of the lifetime of each component. Thus, the time to failure of a series system can be defined in terms of the random variable X as follows:

$$X = \min(Y_1, Y_2, \ldots, Y_Z) \tag{11}$$

Probability density function $f_Y(x;\boldsymbol{\theta})$ and cumulative distribution function notated by $F_Y(x;\theta)$ used the order statistics concept and the compounding method derived by Lu and Shi (2022). The cdf of $X|Z$ or random variable X conditional Z component can be defined with:

$$
\begin{aligned}
F(x|z;\boldsymbol{\theta}) &= P(X \leq x|Z = z) \\
&= 1 - P(X > x|Z = z) \\
&= 1 - P(Y_1 > x)P(Y_2 > x) \dots P(Y_z > x) \\
&= 1 - [S_Y(x;\boldsymbol{\theta})]^z \\
F(x|z;\boldsymbol{\theta}) &= 1 - [1 - F_Y(x;\boldsymbol{\theta})]^z
\end{aligned} \tag{12}
$$

From **Equation (12)** we can derive the probability density function by doing differentiation of $X|Z$ toward x.

$$
f(x|z;\boldsymbol{\theta}) = \frac{d}{dx}F_{X|Z}(x;\boldsymbol{\theta}) = \frac{d}{dx}[1 - (1 - F_Y(x;\boldsymbol{\theta})]^{z-1} \tag{13}
$$

Random variable Z are ZT-Poisson, hence we can find the joint pdf of X and Z as follows:

$$
\begin{aligned}
f_{X,Z}(x,z;\lambda,\boldsymbol{\theta}) &= f_Z(z;\lambda)f_{X|Z}(x|z;\boldsymbol{\theta}) \\
&= \frac{e^{-\lambda}\lambda^z}{\Gamma(z+1)(1-e^{-\lambda})} z f_Y(x;\boldsymbol{\theta})[1 - F_Y(x;\boldsymbol{\theta})]^{z-1} \\
&= \frac{e^{-\lambda}\lambda^{z-1}\cdot\lambda}{z\Gamma(z)(1-e^{-\lambda})} z f_Y(x;\boldsymbol{\theta})[1 - F_Y(x;\boldsymbol{\theta})]^{z-1} \\
&= \frac{\lambda\cdot e^{-\lambda}}{(1-e^{-\lambda})} f_Y(x;\boldsymbol{\theta}) \frac{\lambda^{z-1}[1 - F_Y(x;\boldsymbol{\theta})]^{z-1}}{\Gamma(z)}
\end{aligned}
$$

Next, we can find the marginal probability for X as follow:

$$
f_X(x;\lambda,\boldsymbol{\theta}) = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} f_Y(x;\boldsymbol{\theta}) \sum_{z=0}^{\infty} \frac{\lambda^{z-1}[1 - F_Y(x;\boldsymbol{\theta})]^{z-1}}{\Gamma(z)}
$$

Note that the summation in the equation above is the Taylor series form for $e^{\lambda(1-F_Y(x;\theta))}$, so the equation can be rewritten as:

$$
f(x,\lambda,\theta) = \frac{\lambda}{(1-e^{-\lambda})} f_Y(x;\boldsymbol{\theta}) e^{-\lambda(F_Y(x;\boldsymbol{\theta}))} \tag{14}
$$

3.3 Deriving Cumulative Distribution Function Compounding Method with ZT-Poisson

By doing integral of **Equation (14)** we can find the cumulative distribution function

$$
F_X(x;\lambda,\boldsymbol{\theta}) = \Pr(X \leq x) = \int_0^x \frac{\lambda}{(1-e^{-\lambda})} f_Y(x;\boldsymbol{\theta}) e^{-\lambda(F_Y(x;\boldsymbol{\theta}))} = \frac{1 - e^{-\lambda F_Y(x;\boldsymbol{\theta})}}{1 - e^{-\lambda}} \tag{15}
$$

3.4 Probability density function of Weibull-Poisson Distribution

In 2012, Wanbo Lu and Daimin Shi introduced the Weibull-Poisson distribution. This distribution is obtained through the ZT-Poisson compounding method, as described in the previous section by using the Weibull distribution as a random variable. The Weibull distribution is a generalized form of the Exponential distribution, so it is reasonable to replace the compound distribution in the Exponential-Poisson distribution by Kus from the Exponential distribution with the Weibull distribution.

Suppose there is a random variable that has the following conditions:

$$X = \min(Y_1, Y_2, \dots, Y_Z)$$

Where Z is a ZT-Poisson distributed random variable with parameters λ and $\{Y_i\}_{i=1}^{Z}$ is an identical and independent Weibull distributed random variable with parameters α and β. From **Equation (9)**, the pdf random variable X is as follows:

$$f(x; \alpha, \beta, \lambda) = \frac{\alpha\beta\lambda x^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x^{\alpha}+\lambda \exp(-\beta x^{\alpha})} \text{ with } \alpha, \beta, x > 0 \text{ and } \lambda = 1,2, \dots \quad (16)$$

Some of the characteristics of the Weibull-Poisson distribution pdf are:

1. The Weibull-Poisson distribution will become an Exponential-Weibull distribution when $\alpha = 1$.
2. When λ approaches the value 0 or λ→0, the Weibull-Poisson distribution becomes the Weibull distribution with two parameters.
3. For values $0 < \alpha \leq 1, f(x)$ will be monotonically descending from the initial value x onwards. This can be seen from the blue, red and yellow lines (in **Figure 1**) that never cross the x-axis.
4. For values $\alpha > 1, f(x)$ will be monotonically increasing at the beginning of the x value to a certain point and then monotonically decreasing for the larger x value. In other words, f(x) is unimodal and has only one maximum value at (0, ∞).

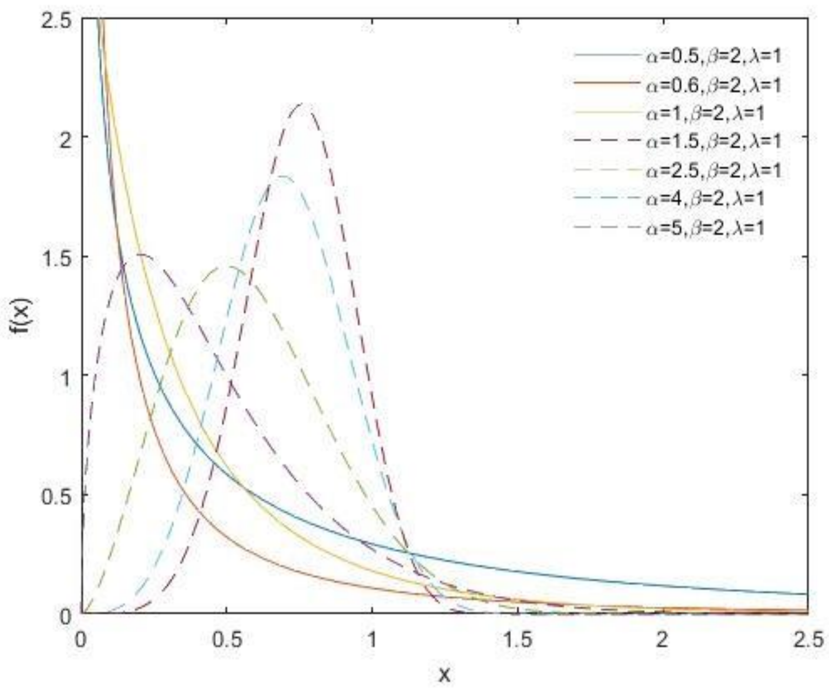

Figure 1. Pdf of Weibull-Poisson Distribution with several values of α and β

3.5 Cumulative Distribution Function of Weibull-Poisson

From doing integration of **Equation (16)** we can get the cdf of Weibull-Poisson.

$$F_X(x) = \frac{1}{1-e^{\lambda}} \left(e^{\lambda \exp(-\beta x^{\alpha})} - e^{\lambda}\right) \quad (17)$$

3.6 Survival Function of Weibull-Poisson Distribution

The survival function of the Weibull-Poisson distribution is obtained by $S_X(x) = 1 - F_X\ (x)$ **[12] [13]**

$$S_X(x) = \frac{1 - e^{\lambda \exp(-\beta x^{\alpha})}}{1 - e^{\lambda}} \tag{18}$$

3.7 Hazard Function of Weibull-Poisson Distribution

Hazard function of the Weibull-Poisson distribution with the probability density function and the survival function of the Weibull-Poisson distribution in **Equation (16)** and **Equation (18)**.

$$h_X(x) = \frac{f_X(x)}{S(x)} = \frac{\frac{\alpha\beta\lambda x^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x^{\alpha}+\lambda \exp(-\beta x^{\alpha})}}{\frac{1-e^{\lambda \exp(-\beta x^{\alpha})}}{1-e^{\lambda}}}$$

$$= \frac{\alpha\beta\lambda x^{\alpha-1} \cdot e^{-\lambda-\beta x^{\alpha}+\lambda\, exp(-\beta x^{\alpha})} \cdot (1-e^{-\lambda})}{(1-e^{-\lambda})(1-e^{\lambda\, exp(-\beta x^{\alpha})})}$$

$$= \frac{\alpha\beta\lambda x^{\alpha-1}\left(1-e^{-\lambda}\right) \cdot e^{-\lambda-\beta x^{\alpha}+\lambda\, exp(-\beta x^{\alpha})}}{(1-e^{-\lambda})(1-e^{\lambda\, exp(-\beta x^{\alpha})})}$$

$$= \frac{\alpha\beta\lambda x^{\alpha-1} \cdot e^{-\lambda-\beta x^{\alpha}+\lambda\, exp(-\beta x^{\alpha})}}{-1 \cdot (1-e^{\lambda\, exp(-\beta x^{\alpha})})}$$

$$= \frac{\alpha\beta\lambda x^{\alpha-1} \cdot e^{-\lambda-\beta x^{\alpha}+\lambda\, exp(-\beta x^{\alpha})}}{-1 \cdot (1-e^{\lambda\, exp(-\beta x^{\alpha})})}$$

$$= \frac{\alpha\beta\lambda x^{\alpha-1} \cdot e^{-\beta x^{\alpha}+\lambda\, exp(-\beta x^{\alpha})}}{e^{\lambda\, exp(-\beta x^{\alpha})} - 1}$$

So, the hazard function of Weibull-Poisson distribution is:

$$h_X(x) = \frac{\alpha\beta\lambda x^{\alpha-1} e^{-\beta x^{\alpha}+\lambda \exp(-\beta x^{\alpha})}}{e^{\lambda \exp(-\beta x^{\alpha})} - 1} \tag{19}$$

Graph analysis of the hazard function of the Weibull-Poisson distribution with parameters α, β, and λ divided into several possible values of α, β, and λ. Proof to the hazard function can be found in **[14] [15]**.

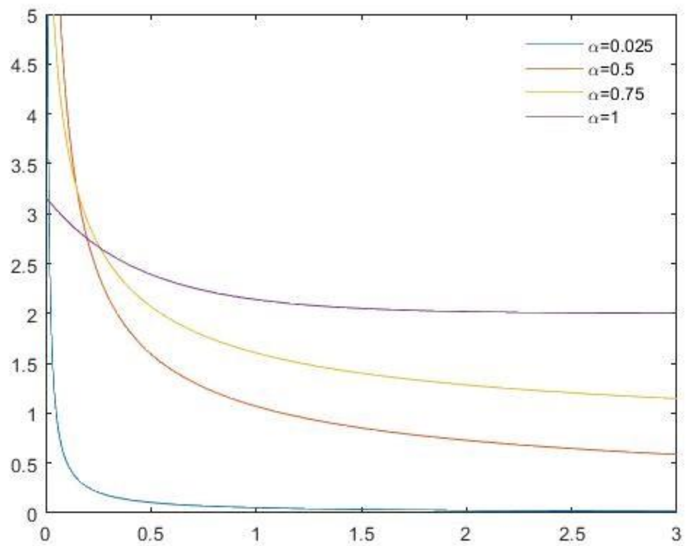

Figure 2. Hazard function of Weibull-Poisson distribution with $\beta = 2$ and $\lambda = 1$

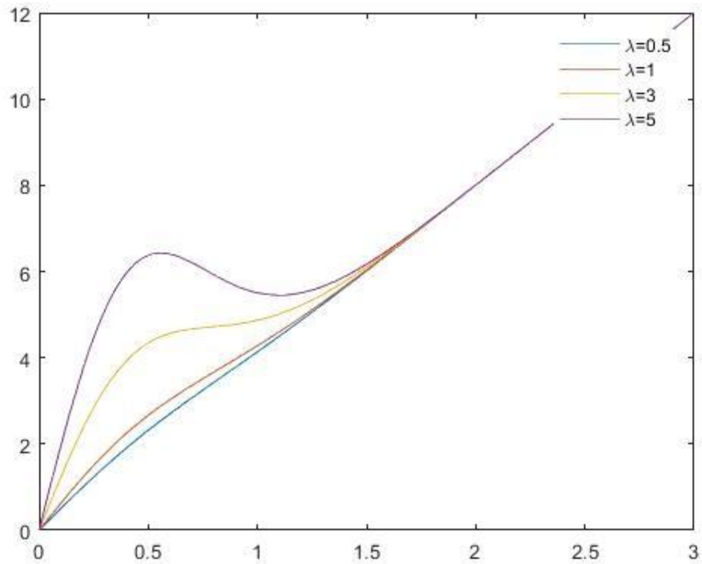

Figure 3. Hazard function of Weibull-Poisson distribution with $\alpha = 2$ and $\beta = 2$

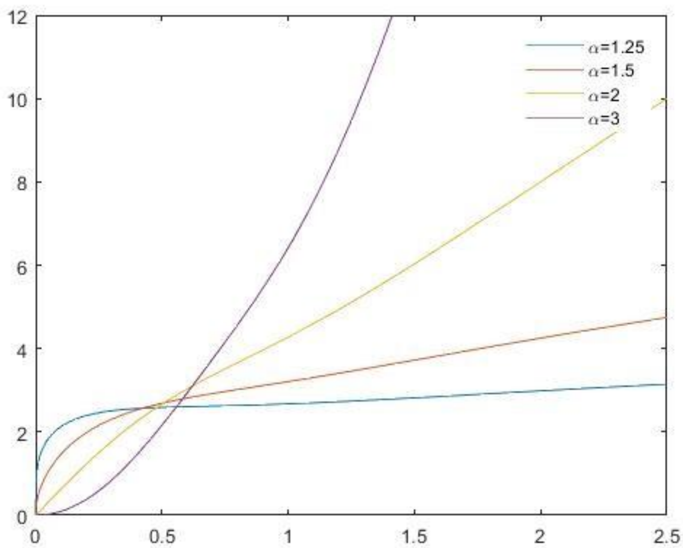

Figure 4. Hazard function of Weibull-Poisson distribution with $\beta = 2$ and $\lambda = 2$

3.8 *R-th* Moment of Weibull-Poisson Distribution

Let X~Weibull Poisson (x; α, β, λ). The form of the rth moment of the Weibull-Poisson distribution is defined as follows:

$$\mu_r' = E(X^r) = \int_0^\infty x^r \cdot f_X(x)dx = \int_0^\infty x^r \cdot \frac{\alpha\beta\lambda x^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x^\alpha+\lambda\, exp(-\beta x^\alpha)}dx$$

$$\mu_r' = \frac{1}{1-e^{-\lambda}}\int_0^\infty x^r \cdot \alpha\beta\lambda x^{\alpha-1} e^{-\lambda-\beta x^\alpha+\lambda\, exp(-\beta x^\alpha)}dx$$

Let's say $y = \lambda e^{-\beta x^\alpha}$. In such way, we get:

$$y = \lambda e^{-\beta x^\alpha} \leftrightarrow \frac{y}{\lambda} = e^{-\beta x^\alpha} \leftrightarrow log\left(\frac{y}{\lambda}\right) = log\left(e^{-\beta x^\alpha}\right) = -\beta x^\alpha \leftrightarrow \frac{log(y) - log(\lambda)}{-\beta} = x^\alpha$$

$$\leftrightarrow -\beta^{-1}(log(y) - log(\lambda)) = x^\alpha \leftrightarrow x^\alpha = [-\beta^{-1}(log(y) - log(\lambda))]^{1/\alpha}$$

With *chain rule* we get $dy = -\lambda\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}dx \leftrightarrow dx = -\frac{dy}{\lambda\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}}$

Integration limit when $x \to 0$, then $y = \lambda$ and when $x \to \infty$, then $y = 0$

$$E(X^r) = \frac{e^{-\lambda}}{1-e^{-\lambda}}\int_\lambda^0 x^r \cdot \alpha\beta\lambda x^{\alpha-1}e^{-\lambda-\beta x^\alpha+\lambda\exp(-\beta x^\alpha)} \frac{dy}{-\lambda\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}}$$

We substitute the value of x with its value in the previous equation

$$= \frac{1}{e^{-\lambda}-1}\int_0^\lambda e^y \cdot [\beta^{-1}(log(y) - log(\lambda))]^{r/\alpha} \cdot dy$$

Then the rth moment of the random variable X is:

$$E(X^r) = \frac{1}{e^{-\lambda}-1}\int_0^\lambda e^y \cdot [\beta^{-1}(log(y) - log(\lambda))]^{r/\alpha} \cdot dy$$

After knowing the shape of the r-th moment of the Weibull-Poisson distribution, we can then determine the first moment or mean (μ), the second moment, the third moment, and the fourth moment of the Weibull-Poisson distribution.

$$E(X) = \frac{1}{e^{-\lambda}-1}\int_0^\lambda e^y \cdot [\beta^{-1}(log(y) - log(\lambda))]^{1/\alpha} \cdot dy$$

$$E(X^2) = \frac{1}{e^{-\lambda}-1}\int_0^\lambda e^y \cdot [\beta^{-1}(log(y) - log(\lambda))]^{\frac{2}{\alpha}} \cdot dy$$

$$E(X^3) = \frac{1}{e^{-\lambda}-1}\int_0^\lambda e^y \cdot [\beta^{-1}(log(y) - log(\lambda))]^{3/\alpha} \cdot dy$$

3.9 Parameter Estimation of the Weibull-Poisson Distribution

The likelihood function of the Weibull-Poisson random variable is

$$L(\alpha,\beta,\lambda|x) = f_x(x_1;\alpha,\beta,\lambda) \cdot f_x(x_2;\alpha,\beta,\lambda) \ldots f_x(x_n;\alpha,\beta,\lambda)$$

$$= \frac{\alpha\beta\lambda x_1^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_1^\alpha+\lambda\, exp(-\beta x_1^\alpha)} \ldots \frac{\alpha\beta\lambda x_n^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_n^\alpha+\lambda\, exp(-\beta x_n^\alpha)}$$

$$= \frac{(\alpha\beta\lambda)^n}{(1-e^{-\lambda})^n} \cdot \prod_{i=1}^n (x_i)^{\alpha-1} \cdot e^{-\lambda\cdot n} \cdot e^{-\beta\Sigma_{i=1}^n x_i} \cdot e^{\lambda\Sigma_{i=1}^n exp(-\beta x_i^\alpha)}$$

$$= \left(\frac{1}{e^\lambda - 1}\right)^n \cdot \prod_{i=1}^n (\alpha\beta\lambda)^n (x_i)^{\alpha-1} \cdot e^{-\lambda\cdot n} \cdot e^{-\beta\Sigma_{i=1}^n x_i} \cdot e^{\lambda\Sigma_{i=1}^n exp(-\beta x_i^\alpha)}$$

To do the estimation, we need to derive the likelihood function, but this equation is a multiplication of several terms, so direct derivation would be difficult to do. So, it takes a logarithmic function so that the results become simpler. The logarithmic function of L(α, β, θ) is denoted as l(α, β, θ) and can be found as follows:

$$l(\alpha,\beta,\lambda|x) = \ln L(\alpha,\beta,\lambda|x)$$

$$= ln\left[\left(\frac{1}{e^{\lambda}-1}\right)^{n} \cdot \prod_{i=1}^{n}(x_i)^{\alpha-1} \cdot (\alpha\beta\lambda)^{n} \cdot e^{-\beta\sum_{i=1}^{n}x_i^{\alpha}} \cdot e^{\lambda\sum_{i=1}^{n}exp\,(-\beta x_i^{\alpha})}\right]$$

$$= \ln\left[\left(\frac{1}{e^{\lambda}-1}\right)^{n}\right] + \ln\left[\prod_{i=1}^{n}(x_i)^{\alpha-1}\right] + \ln[(\alpha\beta\lambda)^{n}] + \ln\left[e^{-\beta\sum_{i=1}^{n}x_i^{\alpha}}\right] + \ln[e^{\lambda\sum_{i=1}^{n}exp\,(-\beta x_i^{\alpha})}]$$

$$= -n\cdot\ln\left(e^{\lambda}-1\right) + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) + n\cdot\ln(\alpha\beta\lambda) - \beta\sum_{i=1}^{n}x_i^{\alpha} + \lambda\sum\nolimits_{i=1}^{n}e^{-\beta x_i^{\alpha}}$$

Let's say $\alpha(x) = \lambda\Sigma_{i=1}^{n}e^{-\beta x_i^{\alpha}}$dan $b(x) = -\beta\Sigma_{i=1}^{n}x_i^{\alpha}$ so the equation can be written as:

$$l(\alpha,\beta,\lambda|x) = n\cdot\ln(\alpha\beta\lambda) + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) + \mathrm{b(x)} + a(x) - n\cdot\ln(e^{\lambda}-1)$$

We can use this equation to obtain the values of the parameters α, β," and" λ in the Weibull-Poisson distribution by deriving the equation and equating the value of the derivative with 0. To make it easier, we will first derive the equations a(x) and b (x) to α.

$$\frac{\delta}{\delta\alpha}a(x) = \frac{\delta}{\delta\alpha}\left[\lambda\sum_{i=1}^{n}e^{-\beta x_i^{\alpha}}\right] = \lambda\cdot\frac{\delta}{\delta\alpha}\left[e^{-\beta x_1^{\alpha}} + e^{-\beta x_2^{\alpha}} + \cdots + e^{-\beta x_n^{\alpha}}\right]$$

$$= \lambda\left[e^{-\beta x_1^{\alpha}}\cdot(-\beta)\cdot ln(x_1)\cdot(x_1^{\alpha}) + \cdots + e^{-\beta x_n^{\alpha}}\cdot(-\beta)\cdot ln(x_n)\cdot(x_n^{\alpha})\right]$$

$$= \sum_{i=1}^{n}\ln(x_1)\cdot(\lambda e^{-\beta x_1^{\alpha}})(-\beta x_i^{\alpha})$$

$$\frac{\delta}{\delta\alpha}b(x) = \frac{\delta}{\delta\alpha}[-\beta\Sigma_{i=1}^{n}x_i^{\alpha}] = -\beta\cdot\frac{\delta}{\delta\alpha}[x_1^{\alpha} + \cdots + x_n^{\alpha}]$$

$$= \beta\cdot[\ln x_1\cdot x_1^{\alpha} + \cdots + \ln x_n\cdot x_n^{\alpha}]$$

$$= \sum_{i=1}^{n}\ln(x_1)(-\beta)(x_1^{\alpha})$$

Entering the derivative results in the initial equation, we get:

$$\frac{\delta}{\delta\alpha}l(\alpha,\beta,\lambda|x) = \frac{\delta}{\delta\alpha}\left[n\cdot\ln(\alpha\beta\lambda) + (\alpha-1)\Sigma_{i=1}^{n}\ln(x_i) + a(x) + b(x) - n\cdot\ln\left(e^{\lambda}-1\right)\right]$$

$$= \frac{\delta}{\delta\alpha}[n\cdot ln(\alpha\beta\lambda)] + \frac{\delta}{\delta\alpha}\left[(\alpha-1)\sum_{i=1}^{n}ln(x_i)\right] + \frac{\delta}{\delta\alpha}[a(x)] + \frac{\delta}{\delta\alpha}[b(x)] - \frac{\delta}{\delta\alpha}\left[n\cdot ln\left(e^{\lambda}-1\right)\right]$$

$$= \frac{n}{\alpha} + \sum_{x=0}^{n}\ln(x_i) + \sum_{x=0}^{n}\ln(x_i)\left(\lambda e^{-\beta x_i^{\alpha}}\right)(-\beta x_i^{\alpha}) + \sum\nolimits_{i=1}^{n}\ln(x_i)\,(-\beta x_i^{\alpha}) + \frac{\delta}{\delta\alpha}[b(x)] - \frac{\delta}{\delta\alpha}\left[n\ln\left(e^{\lambda}-1\right)\right]$$

$$= \frac{n}{\alpha} + \sum\nolimits_{x=0}^{n}\ln(x_i)\left[1 - \beta x_i^{\alpha}\left(1 - \lambda e^{-\beta x_i^{\alpha}}\right)\right] = 0$$

So we can estimate the parameter α for Weibull-Poisson

$$\frac{\delta}{\delta\alpha} l(\alpha,\beta,\lambda|x) = \frac{n}{\alpha} + \sum_{x=0}^{n} \ln(x_i)\left[1 - \beta x_i^{\alpha}\left(1 - \lambda e^{\beta x_i^{\alpha}}\right)\right] = 0$$

Now we do partial derivation for the equation β like the previous methods, hence the equation for β is searched and the estimate is obtained:

$$\frac{\delta}{\delta\beta} l(\alpha,\beta,\lambda|x) = \frac{n}{\beta} - \Sigma_{i=1}^{n} x_1^{\alpha}\left(1 + \lambda e^{-\beta x_1^{\alpha}}\right) = 0$$

Now we do partial derivation for the equation λ like the previous methods, hence the equation for λ is searched and the estimate is obtained:

$$\frac{\delta}{\delta\lambda} l(\alpha,\beta,\lambda|x) = \frac{n}{\lambda} - \Sigma_{i=1}^{n} e^{-\beta x_1^{\alpha}\alpha} - n\frac{e^{\lambda}}{e^{\lambda} - 1} = 0$$

3.10 Data Analysis

In this section, the Weibull Poisson distribution will be applied to data. The data used is the survival of a group of guinea pigs that have been infected with the virus Tubercle Bacilli [7], which visualizes systematic parallel risk in this paper. These data on the life span days of 72 guinea pigs after infection.

It is suspected that this data has a unimodal (upside-down bathtub) hazard function monotone rise. This estimate is based on the average lifetime (in days) of guinea pigs after being infected the same, so there will be a point where the function hazard will be higher than other points. In addition, the increased hazard function is also predictable due to the limited life span of guinea pigs. Besides that, because this data is lifetime data, it can be expected to be modeled in the Weibull-Poisson distribution. The exploratory data analysis is as follows:

Table 1. Descriptive Statistics of Guinea Pig Survival Data

Criteria	Value
Mean	142
Sample Variance	11,931.11
Std. Deviation	109.229
Minimum	43
Maximum	598
Range	555
Sum of all Life Span Days	10,299
Number of Sample	72

This paper has discussed another lifetime distribution, namely the Weibull distribution. Because of this, apart from being modeled with the Weibull-Poisson distribution, data guinea pig survival will also be modeled into the Weibull distribution. The first step is to estimate the parameters of each distribution for the data, and then the distribution compatibility test will be carried out using the Kolmogorov-Smirnov test. Finally, we will look at the shape of the hazard function for the Weibull-Poisson distribution, which has parameters in accordance with guinea pig survival data. Parameter estimation of the Weibull-Poisson distribution with the maximum likelihood method are:

$$\hat{\alpha} = 1.3125\ \hat{\beta} = 0.000291\ \hat{\lambda} = 4.3443$$

Parameter estimation of the Weibull distribution with the maximum method likelihoods are:

$$\hat{\alpha} = 1.4581\ \hat{\beta} = 0.000423$$

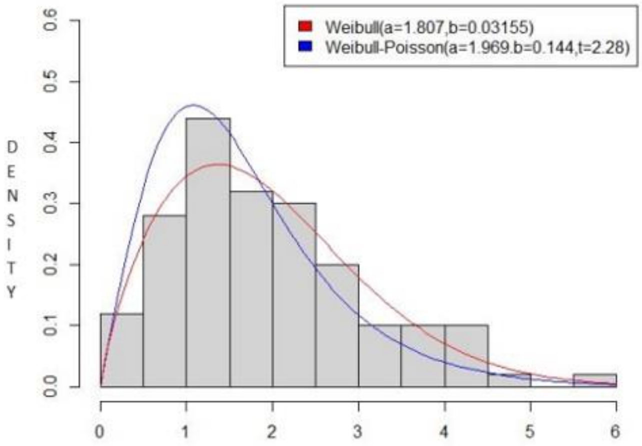

Figure 5. Histogram and pdf plot of Weibull and Weibull-Poisson Distribution

Figure 5 shows that the Weibull-Poisson distribution is more similar the form of the data in with a higher maximum peak value when compared to the probability density function of the Weibull distribution.

A summary of the results of the MLE assessment and the Kolmogorov-Smirnov test is shown in Table 2 below:

Table 2. Estimated Parameter and Kolmogorov-Smirnov Test

Distribution	Level			Kolmogorov Smirnov
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	
Weibull-Poisson	1.3215	0.00029	-53.4933	0.169405
Weibull	1.45815	0.000423	--	0.277337

After conducting a model fit test using the Kolmogorov-Smirnov on guinea pig survival data on the Weibull distribution and Weibull-Poisson distribution, it was concluded that the Weibull distribution was not suitable for modeling survival data on guinea pigs, but the Weibull-Poisson distribution is suitable for modeling survival data guinea pigs from the estimated parameters of each distribution to the data.

4. CONCLUSIONS

The Weibull-Poisson distribution is obtained by compounding the Weibull distribution and the Zero-Truncated Poisson distribution. This compounding method models the lifetime of a series system with Z components where Z is a Poisson random variable and Yi is a Weibull random variable, which describes the lifetime of each component. This distribution has three parameters, namely α, β and λ. In addition, the Weibull Poisson distribution can describe failures in series systems caused by Z possible components, and Yi is the failure time of each component. Where the hazard function can be in the form of monotone down, monotone up, and upside-down bathtub.

The shape of the probability density function of the Weibull-Poisson distribution is determined by the value of the parameter α as follows:

- The density function is unimodal for $\alpha > 1$
- The density function is monotonically decreasing for $0 < \alpha \leq 1$

The form of the hazard function is determined by the alpha and lambda parameters as follows:

- The monotonic hazard function decreases for $\alpha < 1$
- The hazard function is monotonically increasing for $\alpha > 1$ and $\lambda \leq \alpha - 1$
- The hazard functions are monotonically rising and upside-down bathtubs for $\alpha > 1$ and $\lambda>\alpha/(\alpha-1)$

In the data presented, the results of the Kolmogorov-Smirnov test prove that the Weibull-Poisson distribution can model data with a unimodal hazard form (upside-down bathtub) that monotonically increases.

Parameter estimation using the maximum likelihood method cannot provide an analytical solution so that the help of the Newton-Rapson numerical method is needed to obtain a solution from the estimated parameter values of the Weibull-Poisson distribution. The Newton-Rapson numerical method is carried out with the help of nlminb, optim, and maxLik functions in R version 4.0.0 software.

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