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ON THE COMMUTATION MATRIX

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ABSTRACT

The commutation matrix is a matrix that transforms any vec matrix $A, m \times n$, to vec transpose A. In this article, three definitions of the commutation matrix are presented in different ways. It is shown that these three definitions are equivalent. Proof of the equivalent uses the properties in the Kronecker product on the matrix. We also gave the example of the commutation matrix using three ways as Moreover, in this study, we investigate the properties of the commutation matrix related to its transpose and the relation between the vec matrix and the vec transpose matrix using the commutation matrix. We have that the transpose and the inverse of the commutation matrix is its transpose.

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1. INTRODUCTION

The vec matrix is a unique operation that can change a matrix into a column vector [1]. It can also be said to change the matrix into a vector by stacking the column vertically [2]. Note that $vec(A)$ and $vec(A^T)$ have duplicate entries, but the composition of the elements is different. If the matrix A, $m \times n$, and its transpose is matrix A^T, then the vectors $vec(A)$ and $vec(A^T)$ are $mn \times 1$. The vec operator then inspires the creation of new operators with similar transformations, making the matrix a column vector. Operators who were later born were $vech$, $vecd$, $vecp$, and $vecb$ operators (see [3], [4], [5], [6], and [7]). With the new operators after the vec operator, the relation between each operator is also created.

A unique matrix that transforms $vec(A)$ to $vec(A^T)$ for any matrix $m \times n$ [2] is called by commutation matrix. This matrix is defined as a square matrix containing only zeroes and ones. In the previous study, the commutation matrix can be connected to the statistics, i.e., the matrix is applied to some problems related to normal distributions [8]. Furthermore, commutation matrix establishes the relation between the Kronecker product and the *vec*-permutation matrix [9]. Then, [10] extends the concept of the commutation matrix to the commutation tensor and uses the commutation tensor to achieve the unification of the two formulas of linear preserver of the matrix rank.

In [11], it is stated that there are matrices that are like a commutation matrix, i.e., the matrix transform the vec matrix A to the vec transpose matrix A ($A \in \mathbb{C}^{m\times n}$) for the matrices in the Kronecker quaternion group found in [12]. Similar properties are duplicate entries on matrix A in the Kronecker quaternion group with the same position.

The previous study obtained the commutation matrix result using properties of *vec* some matrices (two or more). In this paper, we present the proof of some properties of the commutation matrix differently.

The organization of this paper is as follows. In the Research Methods, some basic concepts and notations of vec, permutation matrix, Kronecker product, and commutation matrix will be used in the section Result and Discussion, are presented. In the section Result and Discussion, the definitions of commutation matrix are discussed. Next, it presents the properties of the commutation matrix.

2. RESEARCH METHODS

The research methods are based on the literature study related to the commutation matrix. The first step of this research is to show the equivalence of the three definitions and then some theorems related to the commutation matrix.

First, this section presents some definitions, properties and theorems related to commutation matrix, i.e., *vec*, permutation matrix, and Kronecker product.

Definition 1 [13] Let $A = [a_{ij}]$ be an $m \times n$ matrix, and A_j is the column of A. The $vec(A)$ is the n-column vector, i.e.,

$$vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

Example 1. Let $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 8 & 0 & 3 & 1 \\ 1 & 4 & 9 & 0 \end{bmatrix}$. Then $vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$, where $A_1 = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.
Or it can also be written with $vec(A) = [1\ 8\ 1\ 3\ 0\ 4\ 1\ 3\ 9\ 3\ 1\ 0]^T$.

Let S_n denote the set of all permutations of the n element set $[n] \coloneqq \{1,2,\dots,n\}$. A permutation is a one-to-one function from $[n]$ onto $[n]$. The permutation of finite sets is usually given by listing each element of the domain and its corresponding functional value.

Example 2. Define a permutation σ of the set $[n] \coloneqq \{1,2,3,4,5,6,7,8\}$ by specifying $\sigma(1) = 7$, $\sigma(2) = 1$, $\sigma(3) = 3$, $\sigma(4) = 6$, $\sigma(5) = 2$, $\sigma(6) = 4$, $\sigma(7) = 5$, $\sigma(8) = 8$. A more convenient way to express this correspondence is to write σ in array form as

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 3 & 6 & 2 & 4 & 5 & 8 \end{bmatrix} \quad (1)$$

There is another notation commonly used to specify permutation. It is called cycle notation. Cycle notation has theoretical advantages in that specific essential properties of the permutation can be readily determined when cycle notation is used. For example, permutation in **Equation (1)** can be written as $\sigma = (1\ 7\ 5\ 2)(4\ 6)$. For detail, see **[14]**.

If σ is a permutation, we have σ change the identity matrix as follows:

Definition 2. [15] Let σ be a permutation in S_n. Define the permutation matrix $P(\sigma) = [\delta_{i,\sigma(j)}]$, $\delta_{i,\sigma(j)} = ent_{i,j}(P(\sigma))$, where

$$\delta_{i,\sigma(j)} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j) \end{cases}.$$

Theorem 1. [14] Let π *and* σ *be two permutations in* S_n, *then* $P(\pi)P(\sigma) = P(\pi\sigma)$.

Definition 3. [1] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. The Kronecker product of A and B, is denoted by $A \otimes B \in \mathbb{R}^{mp\times nq}$ and is defined to be the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Theorem 2. [1] *Let* $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{n\times t}$, *and* $D \in \mathbb{R}^{q\times s}$. *Then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Theorem 3. [1] *Let* $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{n\times t}$, *and* $D \in \mathbb{R}^{q\times s}$. *Then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Theorem 4. [13] *Let* $A, B \in \mathbb{R}^{m\times n}$. *Then* $(A \otimes B)^T = A^T \otimes B^T$.

Theorem 5. [13] *Let* $\boldsymbol{a} \in \mathbb{R}^{m\times 1}$ *and* $\boldsymbol{b} \in \mathbb{R}^{n\times 1}$. *Then* $\boldsymbol{a} \otimes \boldsymbol{b}^T = \boldsymbol{b}^T \otimes \boldsymbol{a}$.

Theorem 6. [16] *Let* $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times q}$. *Then*

$$vec(ABC) = (C^T \otimes A)\, vec(B).$$

Let A be a arbitrary $m \times n$ matrix, the commutation matrix of A is a matrix that transform vec matrix A to the vec transpose matrix. There are several ways to define this matrix, and in this paper is given three different ways to determine this commutation matrix.

Definition 4.

a. **[13]** Let H_{ij} be an $m \times n$ matrix with 1 in its $(i,j)^{th}$ position and zero elsewhere. Then the $mn \times mn$ commutation matrix, denoted by K_{mn}, is given *by:*

$$K_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} (H_{ij} \otimes H_{ij}{}^T)$$

Remark. The matrix H_{ij} can be conveniently expressed using the column from the identity matrices I_m and I_n. If $\boldsymbol{e}_{i,m}$ is the i^{th} column of I_m, and $\boldsymbol{e}_{j,n}$ is the j^{th} column of I_n, then $H_{ij} = \boldsymbol{e}_{i,m}\boldsymbol{e}_{j,n}{}^T$.

b. **[9]** Let I_n be the identity matrix, and $\boldsymbol{e}_{im}$ is an m –dimensional column vector that has 1 in the i^{th} position and 0's elsewhere; that is:

$$\boldsymbol{e}_{i,m} = [0, 0, \dots, 0, 1, 0, \dots, 0]^T \text{ and } I_n \otimes \boldsymbol{e}_{i,m}{}^T = a_{ij}\boldsymbol{e}_{i,m}{}^T,\ a_{ij} \in I_n.$$

The commutation matrix, denoted by $K_{m,n}$ is given by*:*

$$K_{m,n} = \begin{pmatrix} I_n \otimes \boldsymbol{e}_{1m}{}^T \\ I_n \otimes \boldsymbol{e}_{2m}{}^T \\ \vdots \\ I_n \otimes \boldsymbol{e}_{mm}{}^T \end{pmatrix}.$$

c. **[10]** A permutation matrix P is called a commutation matrix of a matrix, $m \times n$, if it satisfies the following conditions:

i. $P = [A_{ij}]$ is an $m \times n$ block matrix, with each block A_{ij} being an $n \times m$ matrix.

ii. For each $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,n\}$, $A_{ij} = \left(a_{st}{}^{(i,j)}\right)$ is a $(0,1)$ matrix with a unique 1 which lies at the position (j,i).

We denote this commutation matrix by $K_{m,n}$, thus a commutation matrix is of size $mn \times mn$.

Example 3. Using **Definition 4** (a – c), we demonstrate how to create the commutation matrix.

Let A be a matrix with size 2×4, then the commutation matrix for A is defined as a matrix 8×8, symbolized by $K_{2,4}$ and using **Definition 4** (a), and **Theorem 5** we have $K_{2,4}$ is:

$$K_{2,4} = \sum_{i=1}^{2}\sum_{j=1}^{4}\left(H_{ij} \otimes H_{ij}{}^{T}\right)$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{4}\left(\left(\boldsymbol{e}_{i,2}\boldsymbol{e}_{j,4}{}^{T}\right)\otimes\left(\boldsymbol{e}_{j,4}\boldsymbol{e}_{i,2}{}^{T}\right)\right)$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{4}\left(\left(\boldsymbol{e}_{i,2}\otimes\boldsymbol{e}_{j,4}\right)\left(\boldsymbol{e}_{j,4}\otimes\boldsymbol{e}_{i,2}\right)^{T}\right)$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{4}\left(\left(\boldsymbol{e}_{i,2}\otimes\boldsymbol{e}_{j,4}\right)\otimes\left(\boldsymbol{e}_{j,4}\otimes\boldsymbol{e}_{i,2}\right)^{T}\right)$$

$$= \sum_{i=1}^{2}\left(e_{i,2}\otimes\sum_{j=1}^{4}\left(\boldsymbol{e}_{j,4}\otimes\boldsymbol{e}_{j,4}{}^{T}\right)\otimes\boldsymbol{e}_{i,2}{}^{T}\right)$$

$$= \sum_{i=1}^{2}\left(e_{i,2}\otimes I_4\otimes\boldsymbol{e}_{i,2}{}^{T}\right)$$

$$= \left(e_{1,2}\otimes I_4\otimes\boldsymbol{e}_{1,2}{}^{T}\right) + \left(e_{2,2}\otimes I_4\otimes\boldsymbol{e}_{2,2}{}^{T}\right)$$

$$= \left(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\end{bmatrix}\right) + \left(\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\otimes\begin{bmatrix}0&1\end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 1\begin{bmatrix}1[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0]\\ 0[1\ \ 0] & 1[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0]\\ 0[1\ \ 0] & 0[1\ \ 0] & 1[1\ \ 0] & 0[1\ \ 0]\\ 0[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0] & 1[1\ \ 0]\end{bmatrix}\\ 0\begin{bmatrix}1[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0]\\ 0[1\ \ 0] & 1[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0]\\ 0[1\ \ 0] & 0[1\ \ 0] & 1[1\ \ 0] & 0[1\ \ 0]\\ 0[1\ \ 0] & 0[1\ \ 0] & 0[1\ \ 0] & 1[1\ \ 0]\end{bmatrix}\end{bmatrix}\right) + \left(\begin{bmatrix} 0\begin{bmatrix}1[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1]\\ 0[0\ \ 1] & 1[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1]\\ 0[0\ \ 1] & 0[0\ \ 1] & 1[0\ \ 1] & 0[0\ \ 1]\\ 0[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1] & 1[0\ \ 1]\end{bmatrix}\\ 1\begin{bmatrix}1[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1]\\ 0[0\ \ 1] & 1[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1]\\ 0[0\ \ 1] & 0[0\ \ 1] & 1[0\ \ 1] & 0[0\ \ 1]\\ 0[0\ \ 1] & 0[0\ \ 1] & 0[0\ \ 1] & 1[0\ \ 1]\end{bmatrix}\end{bmatrix}\right)$$

$$= \begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}.$$

Using **Definition 4** (b), matrix $K_{2,4}$ is a 8×8 permutation matrix partitioned by a 2×4 block matrix, i.e:

$$K_{2,4} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{bmatrix},$$

where $A_{ij} = \left(a_{st}{}^{(i,j)}\right)$ is a 4×2 matrix whose unique non-zero entry is $a_{ji}{}^{(i,j)} = 1$. Specifically

$$K_{2,4} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}.$$

Using **Definition 4** (c), matrix $K_{2,4}$ is defined as:

$$K_{2,4} = \begin{bmatrix} I_4 \otimes \boldsymbol{e}_{12}{}^T \\ I_4 \otimes \boldsymbol{e}_{22}{}^T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} \otimes [1 \quad 0] \\ \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} \otimes [0 \quad 1] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1[1 \quad 0] & 0[1 \quad 0] & 0[1 \quad 0] & 0[1 \quad 0] \\ 0[1 \quad 0] & 1[1 \quad 0] & 0[1 \quad 0] & 0[1 \quad 0] \\ 0[1 \quad 0] & 0[1 \quad 0] & 1[1 \quad 0] & 0[1 \quad 0] \\ 0[1 \quad 0] & 0[1 \quad 0] & 0[1 \quad 0] & 1[1 \quad 0] \end{bmatrix} \\ \begin{bmatrix} 1[0 \quad 1] & 0[0 \quad 1] & 0[0 \quad 1] & 0[0 \quad 1] \\ 0[0 \quad 1] & 1[0 \quad 1] & 0[0 \quad 1] & 0[0 \quad 1] \\ 0[0 \quad 1] & 0[0 \quad 1] & 1[0 \quad 1] & 0[0 \quad 1] \\ 0[0 \quad 1] & 0[0 \quad 1] & 0[0 \quad 1] & 1[0 \quad 1] \end{bmatrix} \end{bmatrix}$$

$$K_{2,4} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}.$$

3. RESULTS AND DISCUSSION

In this section, we present the properties of the commutation matrix. Since, there are three ways to determine the commutation matrix, so in **Theorem 7** is given that the three definitions are equivalent. Furthermore, in **Theorem 8**, is proven that the commutation matrix is the same as its transpose. And last, in **Theorem 9** it is proven that relation between vec matrix and vec transpose matrix.

Theorem 7. *Let $K_{m,n}$ be a commutation matrix. Then the following statements are equivalent:*

1. ***Definition 4*** *(a),*
2. ***Definition 4*** *(b),*
3. ***Definition 4*** *(c).*

Proof. ($1 \Rightarrow 2$) Using **Theorem 2**, **Theorem 3**, **Theorem 4** and **Theorem 5**., we have:

$$K_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(H_{ij} \otimes H_{ij}{}^T \right),$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\boldsymbol{e}_{i,m}\boldsymbol{e}_{j,n}{}^{T}\right)\otimes\left(\boldsymbol{e}_{j,n}\boldsymbol{e}_{i,m}{}^{T}\right)\right),$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\boldsymbol{e}_{i,m}\otimes\boldsymbol{e}_{j,n}\right)\left(\boldsymbol{e}_{j,n}\otimes\boldsymbol{e}_{i,m}\right)^{T}\right),$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\boldsymbol{e}_{i,m}\otimes\boldsymbol{e}_{j,n}\right)\otimes\left(\boldsymbol{e}_{j,n}\otimes\boldsymbol{e}_{i,m}\right)^{T}\right),$$

$$= \sum_{i=1}^{m}\left(e_{i,m}\otimes\sum_{j=1}^{n}\left(\boldsymbol{e}_{j,n}\otimes\boldsymbol{e}_{j,n}{}^{T}\right)\otimes\boldsymbol{e}_{i,m}{}^{T}\right),$$

$$= \sum_{i=1}^{m}\left(e_{i,m}\otimes I_n\otimes\boldsymbol{e}_{i,m}{}^{T}\right),$$

$$= \left(e_{1,m}\otimes I_n\otimes\boldsymbol{e}_{1,m}{}^{T}\right) + \left(e_{2,m}\otimes I_n\otimes\boldsymbol{e}_{2,m}{}^{T}\right) + \cdots + \left(e_{m,m}\otimes I_n\otimes\boldsymbol{e}_{m,m}{}^{T}\right),$$

$$= \left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\otimes\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\otimes\begin{pmatrix}1&0&\cdots&0\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}\otimes\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\otimes\begin{pmatrix}0&1&\cdots&0\end{pmatrix}\right)$$

$$+\cdots+\left(\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\otimes\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\otimes\begin{pmatrix}0&0&\cdots&1\end{pmatrix}\right),$$

$$= \begin{pmatrix}1\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\\0\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\\\vdots\\0\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\end{pmatrix}\otimes\begin{pmatrix}1&0&\cdots&0\end{pmatrix} + \begin{pmatrix}0\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\\1\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\\\vdots\\0\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\end{pmatrix}\otimes\begin{pmatrix}0&1&\cdots&0\end{pmatrix}$$

$$+\cdots+\begin{pmatrix}0\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\\0\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\\\vdots\\1\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\end{pmatrix}\otimes\begin{pmatrix}0&0&\cdots&1\end{pmatrix},$$

$$
= \begin{pmatrix} \begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix} \\ \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \\ \vdots \\ \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \end{pmatrix} \otimes \begin{pmatrix}1&0&\cdots&0\end{pmatrix} + \begin{pmatrix} \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \\ \begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix} \\ \vdots \\ \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \end{pmatrix} \otimes \begin{pmatrix}0&1&\cdots&0\end{pmatrix}
$$

$$
+ \cdots + \begin{pmatrix} \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \\ \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \\ \vdots \\ \begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix} \end{pmatrix} \otimes \begin{pmatrix}0&0&\cdots&1\end{pmatrix},
$$

$$
= \begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\\0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\\&&\vdots&\\0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \otimes \begin{pmatrix}1&0&\cdots&0\end{pmatrix} + \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\\1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\\&&\vdots&\\0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix} \otimes \begin{pmatrix}0&1&\cdots&0\end{pmatrix}
$$

$$
+ \cdots + \begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\\0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\\&&\vdots&\\1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix} \otimes \begin{pmatrix}0&0&\cdots&1\end{pmatrix},
$$

$$
= \begin{pmatrix}
1(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \dots & 0(1\ \ 0 \cdots 0) \\
0(1\ \ 0 \cdots 0) & 1(1\ \ 0 \cdots 0) & \dots & 0(1\ \ 0 \cdots 0) \\
\vdots & \vdots & \ddots & \vdots \\
0(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \cdots & 1(1\ \ 0 \cdots 0) \\
0(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \dots & 0(1\ \ 0 \cdots 0) \\
0(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \dots & 0(1\ \ 0 \cdots 0) \\
\vdots & \vdots & \ddots & \vdots \\
0(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \cdots & 0(1\ \ 0 \cdots 0) \\
 & \vdots & & \\
0(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \dots & 0(1\ \ 0 \cdots 0) \\
0(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \dots & 0(1\ \ 0 \cdots 0) \\
\vdots & \vdots & \ddots & \vdots \\
0(1\ \ 0 \cdots 0) & 0(1\ \ 0 \cdots 0) & \cdots & 0(1\ \ 0 \cdots 0)
\end{pmatrix}
$$

$$
+ \begin{pmatrix}
0(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \dots & 0(0\ \ 1 \cdots 0) \\
0(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \dots & 0(0\ \ 1 \cdots 0) \\
\vdots & \vdots & \ddots & \vdots \\
0(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \cdots & 0(0\ \ 1 \cdots 0) \\
1(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \dots & 0(0\ \ 1 \cdots 0) \\
0(0\ \ 1 \cdots 0) & 1(0\ \ 1 \cdots 0) & \dots & 0(0\ \ 1 \cdots 0) \\
\vdots & \vdots & \ddots & \vdots \\
0(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \cdots & 1(0\ \ 1 \cdots 0) \\
 & \vdots & & \\
0(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \dots & 0(0\ \ 1 \cdots 0) \\
0(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \dots & 0(0\ \ 1 \cdots 0) \\
\vdots & \vdots & \ddots & \vdots \\
0(0\ \ 1 \cdots 0) & 0(0\ \ 1 \cdots 0) & \cdots & 0(0\ \ 1 \cdots 0)
\end{pmatrix} + \cdots
$$

$$
+ \begin{pmatrix}
0(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \dots & 0(0\ \ 0 \cdots 1) \\
0(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \dots & 0(0\ \ 0 \cdots 1) \\
\vdots & \vdots & \ddots & \vdots \\
0(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \cdots & 0(0\ \ 0 \cdots 1) \\
0(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \dots & 0(0\ \ 0 \cdots 1) \\
0(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \dots & 0(0\ \ 0 \cdots 1) \\
\vdots & \vdots & \ddots & \vdots \\
0(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \cdots & 0(0\ \ 0 \cdots 1) \\
 & \vdots & & \\
1(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \dots & 0(0\ \ 0 \cdots 1) \\
0(0\ \ 0 \cdots 1) & 1(0\ \ 0 \cdots 1) & \dots & 0(0\ \ 0 \cdots 1) \\
\vdots & \vdots & \ddots & \vdots \\
0(0\ \ 0 \cdots 1) & 0(0\ \ 0 \cdots 1) & \cdots & 1(0\ \ 0 \cdots 1)
\end{pmatrix}
$$

$$
= \begin{pmatrix} I_n \otimes \boldsymbol{e}_{1m}{}^T \\ I_n \otimes \boldsymbol{e}_{2m}{}^T \\ \vdots \\ I_n \otimes \boldsymbol{e}_{mm}{}^T \end{pmatrix}.
$$

(2 ⇒ 3) Consider that

$$
K_{m,n} = \begin{pmatrix} I_n \otimes \boldsymbol{e}_{1m}{}^T \\ I_n \otimes \boldsymbol{e}_{2m}{}^T \\ \vdots \\ I_n \otimes \boldsymbol{e}_{mm}{}^T \end{pmatrix},
$$

$$
= \begin{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes (1 \quad 0 \cdots \quad 0) \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes (0 \quad 1 \cdots \quad 0) \\
\vdots \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes (0 \quad 0 \cdots \quad 1)
\end{pmatrix},
$$

$$
= \begin{pmatrix}
\begin{pmatrix} 1(1 \quad 0 \cdots \quad 0) & 0(1 \quad 0 \cdots \quad 0) & \cdots & 0(1 \quad 0 \cdots \quad 0) \\ 0(1 \quad 0 \cdots \quad 0) & 1(1 \quad 0 \cdots \quad 0) & \cdots & 0(1 \quad 0 \cdots \quad 0) \\ \vdots & \vdots & \ddots & \vdots \\ 0(1 \quad 0 \cdots \quad 0) & 0(1 \quad 0 \cdots \quad 0) & \cdots & 1(1 \quad 0 \cdots \quad 0) \end{pmatrix} \\
\begin{pmatrix} 1(0 \quad 1 \cdots \quad 0) & 0(0 \quad 1 \cdots \quad 0) & \cdots & 0(0 \quad 1 \cdots \quad 0) \\ 0(0 \quad 1 \cdots \quad 0) & 1(0 \quad 1 \cdots \quad 0) & \cdots & 0(0 \quad 1 \cdots \quad 0) \\ \vdots & \vdots & \ddots & \vdots \\ 0(0 \quad 1 \cdots \quad 0) & 0(0 \quad 1 \cdots \quad 0) & \cdots & 1(0 \quad 1 \cdots \quad 0) \end{pmatrix} \\
\vdots \\
\begin{pmatrix} 1(0 \quad 0 \cdots \quad 1) & 0(0 \quad 0 \cdots \quad 1) & \cdots & 0(0 \quad 0 \cdots \quad 1) \\ 0(0 \quad 0 \cdots \quad 1) & 1(0 \quad 0 \cdots \quad 1) & \cdots & 0(0 \quad 0 \cdots \quad 1) \\ \vdots & \vdots & \ddots & \vdots \\ 0(0 \quad 0 \cdots \quad 1) & 0(0 \quad 0 \cdots \quad 1) & \cdots & 1(0 \quad 0 \cdots \quad 1) \end{pmatrix}
\end{pmatrix},
$$

$$
= \begin{pmatrix}
1(1 \quad 0 \cdots \quad 0) & 0(1 \quad 0 \cdots \quad 0) & & 0(1 \quad 0 \cdots \quad 0) \\
0(1 \quad 0 \cdots \quad 0) & 1(1 \quad 0 \cdots \quad 0) & & 0(1 \quad 0 \cdots \quad 0) \\
\vdots & \vdots & & \vdots \\
0(1 \quad 0 \cdots \quad 0) & 0(1 \quad 0 \cdots \quad 0) & & 1(1 \quad 0 \cdots \quad 0) \\
1(0 \quad 1 \cdots \quad 0) & 0(0 \quad 1 \cdots \quad 0) & & 0(0 \quad 1 \cdots \quad 0) \\
0(0 \quad 1 \cdots \quad 0) & 1(0 \quad 1 \cdots \quad 0) & & 0(0 \quad 1 \cdots \quad 0) \\
\vdots & \vdots & \cdots & \vdots \\
0(0 \quad 1 \cdots \quad 0) & 0(0 \quad 1 \cdots \quad 0) & & 1(0 \quad 1 \cdots \quad 0) \\
\vdots & \vdots & & \vdots \\
1(0 \quad 0 \cdots \quad 1) & 0(0 \quad 0 \cdots \quad 1) & & 0(0 \quad 0 \cdots \quad 1) \\
0(0 \quad 0 \cdots \quad 1) & 1(0 \quad 0 \cdots \quad 1) & & 0(0 \quad 0 \cdots \quad 1) \\
\vdots & \vdots & & \vdots \\
0(0 \quad 0 \cdots \quad 1) & 0(0 \quad 0 \cdots \quad 1) & & 1(0 \quad 0 \cdots \quad 1)
\end{pmatrix},
$$

$$
= \begin{pmatrix}
1 \quad 0 \cdots 0 & 0 \quad 0 \cdots 0 & & 0 \quad 0 \cdots 0 \\
0 \quad 0 \cdots 0 & 1 \quad 0 \cdots 0 & & 0 \quad 0 \cdots 0 \\
\vdots & \vdots & & \vdots \\
0 \quad 0 \cdots 0 & 0 \quad 0 \cdots 0 & & 1 \quad 0 \cdots 0 \\
0 \quad 1 \cdots 0 & 0 \quad 0 \cdots 0 & & 0 \quad 0 \cdots 0 \\
0 \quad 0 \cdots 0 & 0 \quad 1 \cdots 0 & & 0 \quad 0 \cdots 0 \\
\vdots & \vdots & \cdots & \vdots \\
0 \quad 0 \cdots 0 & 0 \quad 0 \cdots 0 & & 0 \quad 1 \cdots 0 \\
\vdots & \vdots & & \vdots \\
0 \quad 0 \cdots 1 & 0 \quad 0 \cdots 0 & & 0 \quad 0 \cdots 0 \\
0 \quad 0 \cdots 0 & 0 \quad 0 \cdots 1 & & 0 \quad 0 \cdots 0 \\
\vdots & \vdots & & \vdots \\
0 \quad 0 \cdots 0 & 0 \quad 0 \cdots 0 & & 0 \quad 0 \cdots 1
\end{pmatrix},
$$

$$
= \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}.
$$

We have that $K_{m,n}$ is a matrix $mn \times mn$ with

i. $P = [A_{ij}]$ is a $m \times n$ block matrix with each block A_{ij} being a $n \times m$ matrix.

ii. For each $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,n\}$, $A_{ij} = \left(a_{st}{}^{(i,j)}\right)$ is a $(0,1)$ matrix with a unique 1 which lies at the position (j,i).

$(3 \Rightarrow 1)$ Consider that

$$K_{m,n} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix},$$

where A_{ij} be an $n \times m$ matrix and for each $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,n\}$, $A_{ij} = \left(a_{s,t}{}^{(i,j)}\right)$ *is a* $(0,1)$ matrix with a unique 1 which lies at the position (j,i). Therefore, we have:

$$K_{m,n} = \left(\begin{array}{cccc} \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{matrix} & & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 1 & 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{matrix} & \cdots & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 1 & \cdots & 0 \end{matrix} \\ \vdots & \vdots & & \vdots \\ \begin{matrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{matrix} & & \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 1 \end{matrix} \end{array}\right)$$

$$= \begin{pmatrix} 1H_{11}{}^T & 0H_{11}{}^T & \dots & 0H_{11}{}^T \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \end{pmatrix} + \cdots + \begin{pmatrix} 0H_{11}{}^T & 0H_{11}{}^T & \dots & 1H_{11}{}^T \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \end{pmatrix}$$

$$+ \begin{pmatrix} 0H_{11}{}^T & 0H_{11}{}^T & \dots & 0H_{11}{}^T \\ 1H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \end{pmatrix} + \cdots + \begin{pmatrix} 0H_{11}{}^T & 0H_{11}{}^T & \dots & 0H_{11}{}^T \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 1H_{11}{}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \end{pmatrix}$$

$$+ \cdots + \begin{pmatrix} 0H_{11}{}^T & 0H_{11}{}^T & \dots & 0H_{11}{}^T \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \\ \vdots & \vdots & \ddots & \vdots \\ 1H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \end{pmatrix} + \cdots + \begin{pmatrix} 0H_{11}{}^T & 0H_{11}{}^T & \dots & 0H_{11}{}^T \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 0H_{11}{}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0H_{11}{}^T & 0H_{11}{}^T & \cdots & 1H_{11}{}^T \end{pmatrix}$$

$$= H_{11} \otimes H_{11}{}^T + \cdots + H_{1n} \otimes H_{1n}{}^T + H_{21} \otimes H_{21}{}^T + \cdots + H_{2n} \otimes H_{2n}{}^T + H_{m1} \otimes H_{m1}{}^T + \cdots + H_{mn} \otimes H_{mn}{}^T$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left(H_{ij} \otimes H_{ij}{}^T\right).$$

Thus, we have

$$K_{m,n} = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} \left(H_{ij} \otimes H_{ij}{}^T\right). \qquad \blacksquare$$

Theorem 8. *Let* $K_{m,n}$ *matrix as in* ***Definition 4***, *then* $K_{m,n} = K_{n,m}{}^T$ *and* $K_{m,n}K_{m,n}{}^T = K_{m,n}{}^T K_{m,n} = I_{mn}$.

Proof. Without loss of generality, we use the definition of $K_{m,n}$ as in **Definition 4** (a).

$$K_{m,n}{}^T = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(H_{ij} \otimes H_{ij}{}^T\right)\right)^T$$

$$= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left((\boldsymbol{e}_{i,m}\boldsymbol{e}_{j,n}{}^{T})\otimes(\boldsymbol{e}_{j,n}\boldsymbol{e}_{i,m}{}^{T})\right)\right)^{T}$$

$$= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left((\boldsymbol{e}_{i,m}\otimes\boldsymbol{e}_{j,n})(\boldsymbol{e}_{j,n}\otimes\boldsymbol{e}_{i,m})^{T}\right)\right)^{T}$$

$$= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left((\boldsymbol{e}_{i,m}\otimes\boldsymbol{e}_{j,n})\otimes(\boldsymbol{e}_{j,n}\otimes\boldsymbol{e}_{i,m})^{T}\right)\right)^{T}$$

$$= \left(\sum_{i=1}^{m}\left(e_{i,m}\otimes\sum_{j=1}^{n}(\boldsymbol{e}_{j,n}\otimes\boldsymbol{e}_{j,n}{}^{T})\otimes\boldsymbol{e}_{i,m}{}^{T}\right)\right)^{T}$$

$$= \left(\sum_{i=1}^{m}(e_{i,m}\otimes I_n\otimes\boldsymbol{e}_{i,m}{}^{T})\right)^{T}$$

$$= \left((e_{1,m}\otimes I_n\otimes\boldsymbol{e}_{1,m}{}^{T}) + (e_{2,m}\otimes I_n\otimes\boldsymbol{e}_{2,m}{}^{T}) + \cdots + (e_{m,m}\otimes I_n\otimes\boldsymbol{e}_{m,m}{}^{T})\right)^{T}$$

$$= (e_{1,m}\otimes I_n\otimes\boldsymbol{e}_{1,m}{}^{T}) + (e_{2,m}\otimes I_n\otimes\boldsymbol{e}_{2,m}{}^{T}) + \cdots + (e_{m,m}\otimes I_n\otimes\boldsymbol{e}_{m,m}{}^{T})$$

$$= K_{n,m}.$$

Next, we have that

$$K_{m,n}K_{m,n}{}^{T} = K_{m,n}K_{n,m}$$

$$= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(H_{ij}\otimes H_{ij}{}^{T})\right)\left(\sum_{j=1}^{n}\sum_{i=1}^{m}(H_{ji}\otimes H_{ji}{}^{T})\right)$$

$$= \left((e_{1,m}\otimes I_n\otimes\boldsymbol{e}_{1,m}{}^{T}) + (e_{2,m}\otimes I_n\otimes\boldsymbol{e}_{2,m}{}^{T}) + \cdots + (e_{m,m}\otimes I_n\otimes\boldsymbol{e}_{m,m}{}^{T})\right)\left((e_{1,n}\otimes I_m\otimes\boldsymbol{e}_{1,n}{}^{T}) + (e_{2,n}\otimes I_m\otimes\boldsymbol{e}_{2,n}{}^{T}) + \cdots + (e_{m,m}\otimes I_n\otimes\boldsymbol{e}_{m,m}{}^{T})\right)$$

$$= \left((e_{1,m}\otimes I_n\otimes\boldsymbol{e}_{1,m}{}^{T}) + (e_{2,m}\otimes I_n\otimes\boldsymbol{e}_{2,m}{}^{T}) + \cdots + (e_{m,m}\otimes I_n\otimes\boldsymbol{e}_{m,m}{}^{T})\right)\left((e_{1,m}\otimes I_n\otimes\boldsymbol{e}_{1,m}{}^{T}) + (e_{2,m}\otimes I_n\otimes\boldsymbol{e}_{2,m}{}^{T}) + \cdots + (e_{m,m}\otimes I_n\otimes\boldsymbol{e}_{m,m}{}^{T})\right)$$

$$= \begin{pmatrix} I_m & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_m & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_m \end{pmatrix}$$

$$= I_{mn},$$

and

$$K_{m,n}{}^{T}K_{m,n} = K_{n,m}K_{m,n}$$

$$= \left(\sum_{j=1}^{n}\sum_{i=1}^{m}(H_{ji}\otimes H_{ji}{}^{T})\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n}(H_{ij}\otimes H_{ij}{}^{T})\right)$$

$$= \left(\left(e_{1,n} \otimes I_m \otimes \boldsymbol{e}_{1,n}{}^T \right) + \left(e_{2,n} \otimes I_m \otimes \boldsymbol{e}_{2,n}{}^T \right) + \cdots + \left(e_{m,m} \otimes I_n \otimes \boldsymbol{e}_{m,m}{}^T \right) \right) \left(\left(e_{1,m} \otimes I_n \otimes \boldsymbol{e}_{1,m}{}^T \right) + \left(e_{2,m} \otimes I_n \otimes \boldsymbol{e}_{2,m}{}^T \right) + \cdots + \left(e_{m,m} \otimes I_n \otimes \boldsymbol{e}_{m,m}{}^T \right) \right)$$

$$= \left(\left(e_{1,n} \otimes I_m \otimes \boldsymbol{e}_{1,n}{}^T \right) + \left(e_{2,n} \otimes I_m \otimes \boldsymbol{e}_{2,n}{}^T \right) + \cdots + \left(e_{m,m} \otimes I_n \otimes \boldsymbol{e}_{m,m}{}^T \right) \right) \left(\left(e_{1,m} \otimes I_n \otimes \boldsymbol{e}_{1,m}{}^T \right) + \left(e_{2,m} \otimes I_n \otimes \boldsymbol{e}_{2,m}{}^T \right) + \cdots + \left(e_{m,m} \otimes I_n \otimes \boldsymbol{e}_{m,m}{}^T \right) \right)$$

$$= \begin{pmatrix} I_m & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_m & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_m \end{pmatrix}$$

$$= I_{mn}. \qquad \blacksquare$$

Theorem 9. *Let A be an $n \times n$ matrix. Then $vec(A) = K_{n,n} vec(A^T)$.*

Proof. Consider that

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}_{n\times n}$$

$$= a_{11} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} (1 \;\; 0 \;\; \cdots \;\; 0)_{1\times m} + a_{12} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} (1 \;\; 0 \;\; \cdots \;\; 0) + \cdots + a_{1n} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} (1 \;\; 0 \;\; \cdots \;\; 0)$$
$$+ a_{21} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (0 \;\; 1 \;\; \cdots \;\; 0) + a_{22} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} (0 \;\; 1 \;\; \cdots \;\; 0) + \cdots + a_{2n} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} (0 \;\; 1 \;\; \cdots \;\; 0) + \cdots$$
$$+ a_{n1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (0 \;\; 0 \;\; \cdots \;\; 1) + a_{n2} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} (0 \;\; 0 \;\; \cdots \;\; 1) + \cdots + a_{nn} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} (0 \;\; 0 \;\; \cdots \;\; 1)$$

$$= a_{11} \boldsymbol{e}_{1,n} \boldsymbol{e}_{1,n}{}^T + a_{12} \boldsymbol{e}_{2,n} \boldsymbol{e}_{1,n}{}^T + \cdots + a_{1n} \boldsymbol{e}_{n,n} \boldsymbol{e}_{1,n}{}^T + a_{21} \boldsymbol{e}_{1,n} \boldsymbol{e}_{2,n}{}^T + a_{22} \boldsymbol{e}_{2,n} \boldsymbol{e}_{2,n}{}^T + \cdots + a_{2n} \boldsymbol{e}_{n,n} \boldsymbol{e}_{2,n}{}^T + a_{n1} \boldsymbol{e}_{1,n} \boldsymbol{e}_{n,n}{}^T + a_{n2} \boldsymbol{e}_{2,n} \boldsymbol{e}_{n,n}{}^T + \cdots + a_{nn} \boldsymbol{e}_{n,n} \boldsymbol{e}_{n,n}{}^T$$

$$= a_{11} \boldsymbol{e}_{1,n} \boldsymbol{e}_{1,n}{}^T + a_{12} \boldsymbol{e}_{2,n} \boldsymbol{e}_{1,n}{}^T + \cdots + a_{1n} \boldsymbol{e}_{n,n} \boldsymbol{e}_{1,m}{}^T + a_{21} \boldsymbol{e}_{1,n} \boldsymbol{e}_{2,m}{}^T + a_{22} \boldsymbol{e}_{2,n} \boldsymbol{e}_{2,m}{}^T + \cdots + a_{2n} \boldsymbol{e}_{n,n} \boldsymbol{e}_{2,n}{}^T + \cdots + a_{n1} \boldsymbol{e}_{1,n} \boldsymbol{e}_{n,n}{}^T + a_{n2} \boldsymbol{e}_{2,n} \boldsymbol{e}_{n,n}{}^T + \cdots + a_{nn} \boldsymbol{e}_{n,n} \boldsymbol{e}_{n,n}{}^T$$

$$= a_{11} H_{11}{}^T + a_{12} H_{21}{}^T + \cdots + a_{1n} H_{n1}{}^T + a_{12} H_{12}{}^T + a_{22} H_{22}{}^T + \cdots + a_{2n} H_{n2}{}^T + \cdots + a_{n1} H_{1n}{}^T + a_{n2} H_{2n}{}^T + \cdots + a_{nn} H_{nn}{}^T$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} H_{ij}{}^T$$

Thus,

$$vec(A^T) = vec\left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} H_{ij}{}^T \right)$$
$$= vec\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(a_{ij} \boldsymbol{e}_{j,n} \boldsymbol{e}_{i,n}{}^T \right) \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} vec\left(a_{ij} \boldsymbol{e}_{j,n} a_{ij} \boldsymbol{e}_{i,n}{}^T \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} vec\left(\boldsymbol{e}_{j,n} \boldsymbol{e}_{i,n}{}^T A \boldsymbol{e}_{j,n} \boldsymbol{e}_{i,n}{}^T \right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} vec\left({H_{ij}}^T A {H_{ij}}^T\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} vec\left({H_{ij}}^T A {H_{ij}}^T\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(H_{ij} \otimes {H_{ij}}^T\right) vec(A)$$
$$= K_{n,n} vec(A)$$

Using **Theorem 8**, we conclude that $K_{n,n}vec(A^T) = vec(A)$. ■

4. CONCLUSIONS

This paper establishes some conclusions on the commutation matrix. The results of the properties of the commutation matrix using one of the definitions, the commutation matrix. A different way to prove the properties of the commutation matrix is given. All these obtained conclusions make the theory of commutation matrix more complete.

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