BAREKENG: Journal of Mathematics and Its Applications
December 2023 Volume 17 Issue 4 Page 2081–2094
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol17iss4pp2081-2094

THE STUDY OF ECCENTRICITY SPECTRUM AND ENERGY IN PATH AND CYCLE GRAPHS

Ni Kadek Emik Sapitri[1*], Vira Hari Krisnawati[2]

[1,2]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Brawijaya University Malang, 65145, Jawa Timur, Indonesia*

Corresponding author's e-mail: ** ***emikpitri@student.ub.ac.id*

Article History:

Received: 24th June 2023
Revised: 23rd September 2023
Accepted: 25th October 2023

Keywords:

Eccentricity;
Graph;
Eccentricity Spectrum;
Eccentricity Energy;
Path;
Cycle.
.

ABSTRACT

The eccentricity matrix is one of matrices to represent graphs. The eccentricity matrix is used as a basis for calculating the eccentricity spectrum and energy. This article aims to study the concepts of eccentricity spectrum and energy in simple graphs. For special cases, we also discuss eccentricity spectrum and energy of paths and cycles. All studies in this article focus on providing some examples to facilitate the reader's understanding of the concepts studied. In addition, this article also corrects the mistakes in the lemma about eccentricity spectrum of paths and theorem about eccentricity energy of odd-order cycles from reference articles. Corrections are made by indicating where the errors are in the referenced articles, providing counter examples, correcting inaccurate lemmas and theorems, and giving short proofs. At the end of the article, an open problem is also included to provide an overview of research ideas that can be developed from the concepts of eccentricity spectrum and energy.

How to cite this article:

N. K. E. Sapitri and V. H. Krisnawati., "THE STUDY OF ECCENTRICITY SPECTRUM AND ENERGY IN PATH AND CYCLE GRAPHS," *BAREKENG: J. Math. & App.*, vol. 17, iss. 4, pp. 2081-2094, December, 2023.

1. INTRODUCTION

The concept of graph theory was introduced by Euler in 1736 [1]. A graph is a system of a finite non-empty set of vertices and a set of edges. Two vertices connected by at least one edge are said to be adjacent. The number of vertices in a graph is called order and the number of edges in a graph is called size. A simple graph is a graph that has no more than one edge between two vertices and no edges that start and end at the same vertex (without loops) [2].

Graphs can also be grouped into several graph classes based on their shape. Two classes of graphs that are often encountered are path graphs and cycle graphs. Path is a graph of order n and has size $n-1$. Cycle is a graph that has order m and has size m, where $m \geq 3$ [2]. When viewed from the neighboring elements, a path is a graph whose vertices can be arranged in a linear sequence in such a way that the two vertices are adjacent if they are consecutive in the sequence, and vice versa are not adjacent. Meanwhile, a cycle graph is a graph whose vertices can be arranged in a circular order in such a way that two adjacent vertices are consecutive in order [3]. Thus, path can be formed by deleting one edge of cycle.

Graphs can be represented in the form of sets of vertices and edges, diagrams, or matrices [2]. One of the matrices to represent graphs is the eccentricity matrix proposed by Wang et al. in 2018 [4]. Reference [4] explained that the idea of forming an eccentricity matrix originated from the D_{MAX} matrix proposed by Randić in 2013 [5]. The D_{MAX} is a graph matrix that is built from a distance matrix. The distance matrix is a matrix whose entries represent the distance from every two vertices on the graph, namely the size of the shortest path that connects the two vertices [2]. The ij entry value in the D_{MAX} matrix will have the same value as the ij entry in the corresponding distance matrix if that value is greater than or equal to the smallest value between the largest value in the i-th row and the largest value in the j-th column, and vice versa is zero [5]. The D_{MAX} matrix was then redefined and given a new name as the eccentricity matrix by Wang et al [4]. Several studies regarding the properties of the eccentricity matrix in certain graph classes can be seen in [6]–[10].

The eccentricity matrix is needed as initial information to calculate eccentricity spectrum [4]. The eccentricity spectrum is then used as the basis for calculating the value of eccentricity energy [11]. Several studies regarding the eccentricity of energy in certain graph classes can be seen in [12] and [13]. Research on eccentricity spectrum and eccentricity energy in graph theory still has a great possibility to be developed considering the definition of eccentricity spectrum introduced by Wang et al. in 2018 in [4] and the definition of eccentricity energy introduced by Wang et al. in 2019 in [11]. The concept of eccentricity matrix is also related to the eccentricity spectral radius. Eccentricity spectral radius is the largest eigenvalue of the eccentricity matrix [4]. Several studies related to the eccentricity of the spectral radius can be seen in [14]–[17]. However, this article does not include the study of eccentricity spectral radius.

In order to increase the reader's understanding of the concepts of eccentricity spectrum and eccentricity energy, this article studies these concepts in simple graphs and their properties in the path and cycle graphs. The studies focused on providing some examples which were explained in detail and systematically. Section 2 of this article contains research method that were carried out. Subsection 3.1 studies the concepts related to eccentricity spectrum and eccentricity energy. Subsection 3.2 describes the properties of the eccentricity spectrum and eccentricity energy of path. Subsection 3.3 describes the properties of the eccentricity spectrum and the eccentricity energy of cycle. Section 4 contains the conclusions of this entire studies along with open problems which can be used for further research related to the eccentricity matrix. This article is expected to be a reference in understanding the concepts related to eccentricity spectrum and eccentricity energy and their properties for certain types of graphs, namely paths and cycles.

2. RESEARCH METHODS

This article focuses on studying concepts related to eccentricity spectrum and energy and their properties for certain types of graphs, namely paths and cycles. The preparation of the article is based on literature study. The main references used are [4] and [11]. In this literature study, we provide definitions of the eccentricity of vertices in graphs, eccentricity matrix, eccentricity spectrum, and eccentricity energy, along with their properties especially in paths and cycles. Eight examples are given to make it easier to understand concepts based on definitions and characteristics studied from related literature. All of these examples are explained in detail and systematically. In addition, this article also give some corrections of the lemma related to eccentricity spectrum in paths and theorem related to eccentricity energy in odd-order cycles from [11]. This is because there are small mistakes in proving the lemma and the theorem in [11] that make it incorrect. The corrections provided in this article are giving explanations regarding the location of the intended error, counter examples, and providing the appropriate forms of the lemma and theorem. This article also give some short corrected proofs of these lemma and theorem.

3. RESULTS AND DISCUSSION

This section explains the concepts related to eccentricity spectrum and eccentricity energy in a simple graph G and its properties in path and cycle graph classes.

3.1 Eccentricity Spectrum and Eccentricity Energy

This subsection contains some definitions and examples related to the concept of eccentricity in graph G including matrix, eigenvalues, spectrum, and energy.

Definition 1. [4] Let $G = (V, E)$ be a simple graph that has a set of vertices $V(G)$ and a set of edges $E(G)$. The eccentricity of vertex $u \in V(G)$ written as $e_G(u)$ and defined as:

$$e_G(u) = max\{d(u, v) \mid v \in V(G)\}, \quad (1)$$

where $d(u, v)$ is a distance between u and v.

Example 1. Given graph G as in **Figure 1**.

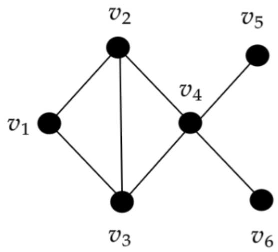

Figure 1. Graph G

Based on **Definition 1** and the definition of distance between two vertices listed in Section 1, it can be obtained:

$$e_G(v_1) = max\begin{Bmatrix} d(v_1, v_1), d(v_1, v_2), d(v_1, v_3), \\ d(v_1, v_4), d(v_1, v_5), d(v_1, v_6) \end{Bmatrix} = \max\{0,1,1,2,3,3\} = 3,$$
$$e_G(v_2) = max\begin{Bmatrix} d(v_2, v_1), d(v_2, v_2), d(v_2, v_3), \\ d(v_2, v_4), d(v_2, v_5), d(v_2, v_6) \end{Bmatrix} = \max\{1,0,1,1,2,2\} = 2,$$
$$e_G(v_3) = max\begin{Bmatrix} d(v_3, v_1), d(v_3, v_2), d(v_3, v_3), \\ d(v_3, v_4), d(v_3, v_5), d(v_3, v_6) \end{Bmatrix} = \max\{1,1,0,1,2,2\} = 2,$$
$$e_G(v_4) = max\begin{Bmatrix} d(v_4, v_1), d(v_4, v_2), d(v_4, v_3), \\ d(v_4, v_4), d(v_2, v_5), d(v_2, v_6) \end{Bmatrix} = \max\{2,1,1,0,1,1\} = 2,$$
$$e_G(v_5) = max\begin{Bmatrix} d(v_5, v_1), d(v_5, v_2), d(v_5, v_3), \\ d(v_5, v_4), d(v_5, v_5), d(v_5, v_6) \end{Bmatrix} = \max\{3,2,2,1,0,2\} = 3,$$
$$e_G(v_6) = max\begin{Bmatrix} d(v_6, v_1), d(v_6, v_2), d(v_6, v_3), \\ d(v_6, v_4), d(v_6, v_5), d(v_6, v_6) \end{Bmatrix} = \max\{3,2,2,1,2,0\} = 3.$$

Definition 2. [4] The eccentricity matrix of graph G is symbolized as $\varepsilon(G)$. The entries in $\varepsilon(G)$ are defined as follows:

$$\varepsilon(G) = \begin{cases} D_{ij} \,; jika\ D_{ij} = min\{e_G(u_i), e_G(u_j)\} \\ 0 \,; jika\ D_{ij} < min\{e_G(u_i), e_G(u_j)\}' \end{cases} \quad (2)$$

where D_{ij} is the entry of i-th row and j-th column of the distance matrix of graph G. In other words, D_{ij} is the distance between vertex v_i and v_j.

Example 2. Given graph G according as in **Figure 1**. Recall that the distance matrix is a symmetric matrix. Consequently, the eccentricity matrix is also a symmetric matrix. Based on **Definition 2** and the results in **Example 1**, we can calculate the eccentricity matrix of graph G as follows:

$$\min\{e_G(v_1), e_G(v_1)\} = \min\{3,3\} = 3\,; D_{11} = 0\,;\ (\varepsilon(G))_{11} = 0,$$

$$\min\{e_G(v_1), e_G(v_2)\} = \min\{3,2\} = 2\,; D_{12} = 1\,;\ \big(\varepsilon(G)\big)_{12} = \big(\varepsilon(G)\big)_{21} = 0,$$
$$\min\{e_G(v_1), e_G(v_3)\} = \min\{3,2\} = 2\,; D_{13} = 1\,;\ \big(\varepsilon(G)\big)_{13} = \big(\varepsilon(G)\big)_{31} = 0,$$
$$\min\{e_G(v_1), e_G(v_4)\} = \min\{3,2\} = 2\,; D_{14} = 2\,;\ \big(\varepsilon(G)\big)_{14} = \big(\varepsilon(G)\big)_{41} = 2,$$
$$\min\{e_G(v_1), e_G(v_5)\} = \min\{3,3\} = 3\,; D_{15} = 3\,;\ \big(\varepsilon(G)\big)_{15} = \big(\varepsilon(G)\big)_{51} = 3,$$
$$\min\{e_G(v_1), e_G(v_6)\} = \min\{3,3\} = 3\,; D_{16} = 3\,;\ \big(\varepsilon(G)\big)_{16} = \big(\varepsilon(G)\big)_{61} = 3,$$
$$\min\{e_G(v_2), e_G(v_2)\} = \min\{2,2\} = 2\,; D_{22} = 0\,;\ \big(\varepsilon(G)\big)_{22} = 0,$$
$$\min\{e_G(v_2), e_G(v_3)\} = \min\{2,2\} = 2\,; D_{23} = 1\,;\ \big(\varepsilon(G)\big)_{23} = \big(\varepsilon(G)\big)_{32} = 0,$$
$$\min\{e_G(v_2), e_G(v_4)\} = \min\{2,2\} = 2\,; D_{24} = 1\,;\ \big(\varepsilon(G)\big)_{24} = \big(\varepsilon(G)\big)_{42} = 0,$$
$$\min\{e_G(v_2), e_G(v_5)\} = \min\{2,3\} = 2\,; D_{25} = 2\,;\ \big(\varepsilon(G)\big)_{25} = \big(\varepsilon(G)\big)_{52} = 2,$$
$$\min\{e_G(v_2), e_G(v_6)\} = \min\{2,3\} = 2\,; D_{26} = 2\,;\ \big(\varepsilon(G)\big)_{26} = \big(\varepsilon(G)\big)_{62} = 2,$$
$$\min\{e_G(v_3), e_G(v_3)\} = \min\{2,2\} = 2\,; D_{33} = 0\,;\ \big(\varepsilon(G)\big)_{33} = 0,$$
$$\min\{e_G(v_3), e_G(v_4)\} = \min\{2,2\} = 2\,; D_{34} = 1\,;\ \big(\varepsilon(G)\big)_{34} = \big(\varepsilon(G)\big)_{43} = 0$$
$$\min\{e_G(v_3), e_G(v_5)\} = \min\{2,3\} = 2\,; D_{35} = 2\,;\ \big(\varepsilon(G)\big)_{35} = \big(\varepsilon(G)\big)_{53} = 2,$$
$$\min\{e_G(v_3), e_G(v_6)\} = \min\{2,3\} = 2\,; D_{36} = 2\,;\ \big(\varepsilon(G)\big)_{36} = \big(\varepsilon(G)\big)_{63} = 2,$$
$$\min\{e_G(v_4), e_G(v_4)\} = \min\{2,2\} = 2\,; D_{44} = 0\,;\ \big(\varepsilon(G)\big)_{44} = 0,$$
$$\min\{e_G(v_4), e_G(v_5)\} = \min\{2,3\} = 2\,; D_{45} = 1\,;\ \big(\varepsilon(G)\big)_{45} = \big(\varepsilon(G)\big)_{54} = 0,$$
$$\min\{e_G(v_4), e_G(v_6)\} = \min\{2,3\} = 2\,; D_{46} = 1\,;\ \big(\varepsilon(G)\big)_{46} = \big(\varepsilon(G)\big)_{64} = 0,$$
$$\min\{e_G(v_5), e_G(v_5)\} = \min\{3,3\} = 3\,; D_{55} = 0\,;\ \big(\varepsilon(G)\big)_{55} = 0,$$
$$\min\{e_G(v_5), e_G(v_6)\} = \min\{3,3\} = 3\,; D_{56} = 2\,;\ \big(\varepsilon(G)\big)_{56} = \big(\varepsilon(G)\big)_{65} = 0,$$
$$\min\{e_G(v_6), e_G(v_6)\} = \min\{3,3\} = 3\,; D_{66} = 0\,;\ \big(\varepsilon(G)\big)_{66} = 0.$$

As a result, the eccentricity matrix of graph G is:

$$\varepsilon(G) = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 0 & 0 & 0 \\ 3 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

The spectrum of a graph is usually formed by the eigenvalues of the adjacency matrices $A(G)$. Thus, an eccentricity spectrum (ε-spectrum) or the spectrum formed from the eccentricity matrix $\varepsilon(G)$ requires ε-eigenvalues or eigenvalues of $\varepsilon(G)$. Since $\varepsilon(G)$ is a symmetric matrix, the ε-eigenvalues of graph G are real.

Definition 3. **[4]** Suppose $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k$ are distinc ε-eigenvalues. The ε-spectrum of graph G can be written as:

$$Spec_\varepsilon(G) = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \cdots & \varepsilon_k \\ m_1 & m_2 & m_3 & \ldots & m_k \end{pmatrix}, \tag{3}$$

where m_i indicating the number of eigenvalues ε_i and $1 \le i \le k$. Furthermore, the largest ε-eigenvalue (ε_1) is called eccentricity spectral radius.

Example 3. Given graph G as in **Figure 1**. In this example, we calculate ε-spectrum of graph G. In **Example 1**, we got:

$$\varepsilon(G) = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 0 & 0 & 0 \\ 3 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Then, we calculate the ε-eigenvalues of $\varepsilon(G)$. The ε-eigenvalues obtained from G are $6.0194, 1.329, 0, 0, -1.329$, and -6.0194. Note that $6.0194 > 1.329 > 0 > -1.329 > -6.0194$. According to **Definition 3**, the ε-spectrum of graph G is:

$$Spec_{\varepsilon}(G) = \begin{pmatrix} 6.0194 & 1.329 & 0 & -1.329 & -6.0194 \\ 1 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

Besides being represented by a spectrum, the eigenvalues of a graph can also be represented by a value called energy. The energy of the eccentricity matrix is called ε-energy.

Definition 4. [11] Suppose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n$ are all ε-eigenvalues of graph G. The eccentricity energy (ε-energy) of graph G is denoted $E_{\varepsilon}(G)$ and defined as:

$$E_{\varepsilon}(G) = \sum_{i=1}^{n} |\varepsilon_i|. \tag{4}$$

Example 4. Given graph G according to **Figure 1**. In this example, we count the ε-energy of graph G. According to **Definition 4** and using the ε-eigenvalues results in **Example 3**, we can get:

$$E_{\varepsilon}(G) = \sum_{i=1}^{n} |\varepsilon_i| = |6.0194| + |1.329| + 2|0| + |-1.329| + |-6.0194| = 14.6968.$$

3.2 The ε-Spectrum and ε-Energy of a Path

This subsection discusses lemmas and examples regarding the properties of the ε-spectrum and ε-energy of paths. **Lemma 1** explains about the properties of ε-spectrum of paths with various order. **Lemma 2** is the correction of errors in **Lemma 1**. Then, **Theorem 1** disscus about the properties of ε-energy of paths with various order.

Lemma 1. [11] *Let P_n be a path with order n.*

(i) If $n = 1,2,3$ then

$$Spec_{\varepsilon}(P_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, Spec_{\varepsilon}(P_2) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$
$$Spec_{\varepsilon}(P_3) = \begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} & -2 \\ 1 & 1 & 1 \end{pmatrix}. \tag{5}$$

(ii) If $n = 2k$ and $k \geq 2$, then

$$Spec_{\varepsilon}(P_{2k}) = \begin{pmatrix} \frac{j+\sqrt{a}}{6} & \frac{j-\sqrt{a}}{6} & \frac{-j+\sqrt{a}}{6} & \frac{-j-\sqrt{a}}{6} & 0 \\ 1 & 1 & 1 & 1 & 2k-4 \end{pmatrix}, \tag{6}$$

where $j = 6k - 3$ and $a = j(14k^2 - 20k + 9)$.

(iii) If $n = 2k + 1$ and $k \geq 2$, then

$$Spec_{\varepsilon}(P_{2k+1}) = \begin{pmatrix} \frac{6k+\sqrt{b}}{6} & \frac{6k-\sqrt{b}}{6} & \frac{-6k+\sqrt{c}}{6} & \frac{-6k-\sqrt{c}}{6} & 0 \\ 1 & 1 & 1 & 1 & 2k-3 \end{pmatrix}, \tag{7}$$

where $b = 6k(14k^2 + 3k + 1)$ and $c = 6k(14k^2 - 9k + 1)$.

Example 5. These examples are related to **Lemma 1**.

(i) For $n = 1$, $n = 2$, and $n = 3$, we can get consecutive graphs P_1, P_2, and P_3 as in **Figure 2**.

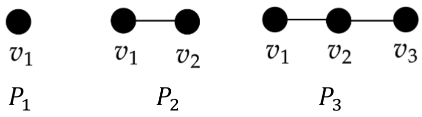

Figure 2. Graphs P_1, P_2, and P_3

According to **Definition 1**, the eccentricity in P_1 is $e_{P_1}(v_1) = \max\{d(v_1, v_1)\} = 0$. Therefore, according to **Definition 2**, $\varepsilon(P_1) = 0$. A zero matrix with size 1×1 has 1 eigenvalue 0. As a result and refer to **Definition 3**, we can get $Spec_{\varepsilon}(P_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Futhermore, the eccentricity of all vertices in P_2 are $e_{P_2}(v_1) = \max\{d(v_1, v_1), d(v_1, v_2)\} = \max\{0,1\} = 1$ and $e_{P_2}(v_2) = \max\{d(v_2, v_1), d(v_2, v_2)\} = \max\{1,0\} = 1$. Thus, according to **Definition 2**, $\varepsilon(P_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Eigenvalues of $\varepsilon(P_2)$ are calculated. The eigenvalues of $\varepsilon(P_2)$ are obtained, namely -1 and 1. Consequently, according to **Definition 3**, we obtain:

$Spec_\varepsilon(P_2) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

In an analogous way, we get $\varepsilon(P_3) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. Thus, the eigenvalues of $\varepsilon(P_3)$ (with 4 decimal places) are $-2, -0.7321,$ and 2.7321. Note that $\sqrt{3} \approx 1.7321$, so $-0.7321 = 1 - \sqrt{3}$ and $2.7321 = 1 + \sqrt{3}$. Thus, according to **Definition 3**, $Spec_\varepsilon(P_3) = \begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} & -2 \\ 1 & 1 & 1 \end{pmatrix}$.

(ii) For n even, choose $k = 4$ so that $n = 2k = 8$. Graph P_8 can be seen on **Figure 3**.

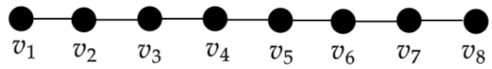

Figure 3. Graph P_8

In a similar way of **Example 1** and **Example 2**, it can be obtained:

$$\varepsilon(P_8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & 5 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\varepsilon(P_8)$ with 4 decimal places are $-12.9472, -5.9472, 0, 0, 0, 0, 5.9472,$ and 12.9472. As a result, according to **Definition 3**, we get:

$$Spec_\varepsilon(P_8) = \begin{pmatrix} 12.9472 & 5.9472 & 0 & -5.9472 & -12.9472 \\ 1 & 1 & 4 & 1 & 1 \end{pmatrix}.$$

Note that the position of the eigenvalues in $Spec_\varepsilon(P_8)$ obtained is not in accordance with **Lemma 1** (ii). So, **Lemma 1** (ii) does not hold.

Claim **Lemma 1** (ii) holds. If $k = 4$, then $j = 6(4) - 3 = 21$ and $a = 21(14(4)^2 - 20(4) + 9) = 3213$, so that:

$$Spec_\varepsilon(P_8) = \begin{pmatrix} \frac{21+\sqrt{3213}}{6} & \frac{21-\sqrt{3213}}{6} & \frac{-21+\sqrt{3213}}{6} & \frac{-21-\sqrt{3213}}{6} & 0 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$
$$\approx \begin{pmatrix} 12.9472 & -5.9472 & 5.9472 & -12.9472 & 0 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}.$$

The writing style of $Spec_\varepsilon(P_8) \approx \begin{pmatrix} 12.9472 & -5.9472 & 5.9472 & -12.9472 & 0 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$ does not match with **Definition 3** because the position of the eigenvalues on the ε-spectrum must be sorted from the largest to the smallest eigenvalue. For this reason, we investigated the connection of j and $\sqrt{a}$ when $k \geq 2$, in order to determine a more appropriate general form of $Spec_\varepsilon(P_{2k})$. We explained the investigation process in short proof of **Lemma 2** (ii).

(iii) For n odd, choose $k = 4$ so that $n = 2k + 1 = 9$. Graph P_9 ilustrated in **Figure 4**.

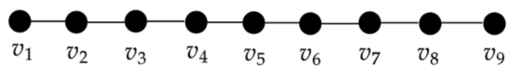

Figure 4. Graph P_9

In a similar way with **Example 1** and **Example 2**, can be obtained:

$$\varepsilon(P_9)=\begin{pmatrix}0&0&0&0&4&5&6&7&8\\0&0&0&0&0&0&0&0&7\\0&0&0&0&0&0&0&0&6\\0&0&0&0&0&0&0&0&5\\4&0&0&0&0&0&0&0&4\\5&0&0&0&0&0&0&0&0\\6&0&0&0&0&0&0&0&0\\7&0&0&0&0&0&0&0&0\\8&7&6&5&4&0&0&0&0\end{pmatrix}$$

Thus, the eigenvalues of $\varepsilon(P_9)$ with 4 decimal places are $-15.2250, -8.5698, 0, 0, 0, 0, 0, 7.2250$, and 16.5698. Thus, according to **Definition 3**, the ε-spectrum of P_9 is:

$$Spec_\varepsilon(P_9)=\begin{pmatrix}16.5698 & 7.2250 & 0 & -8.5698 & -15.2250\\ 1 & 1 & 5 & 1 & 1\end{pmatrix}.$$

Note that the position of the eigenvalues in $Spec_\varepsilon(P_9)$ obtained is not in accordance with **Lemma 1** (iii), so **Lemma 1** (iii) does not hold.

Claim **Lemma 1** (iii) holds. If $k=4$, then $b=6(4)(14(4^2)+3(4)+1)=5688$ and $c=6(4)(14(4^2)-9(4)+1)=4536$, so the ε-spectrum of P_9 is:

$$Spec_\varepsilon(P_9)=\begin{pmatrix}\frac{24+\sqrt{5688}}{6} & \frac{24-\sqrt{5688}}{6} & \frac{-24+\sqrt{4536}}{6} & \frac{-24-\sqrt{4536}}{6} & 0\\ 1 & 1 & 1 & 1 & 5\end{pmatrix}$$
$$\approx\begin{pmatrix}16.5698 & -8.5698 & 7.2250 & -15.2250 & 0\\ 1 & 1 & 1 & 1 & 5\end{pmatrix}.$$

Analogous with (ii) in this example, the writing style of $Spec_\varepsilon(P_9)$ above does not match with **Definition 3**. For this reason, we investigate the connections of $\frac{6k+\sqrt{b}}{6}, \frac{6k-\sqrt{b}}{6}, \frac{-6k+\sqrt{c}}{6}, \frac{-6k-\sqrt{c}}{6}$, and 0, when $k\geq 2$. The goal is to determine the general form of $Spec_\varepsilon(P_{2k+1})$. We explained the investigation process in short proof of **Lemma 2** (iii).

The error in **Lemma 1** is caused by the Lemma proof in **[11]** not considering the order of the eigenvalues before including them in the ε-spectrum form. A following Lemma which is an improvement on **Lemma 1** can be formed as follows:

Lemma 2. *Let P_n be a path with order n.*

(i) If $n=1,2,3$ then

$$Spec_\varepsilon(P_1)=\begin{pmatrix}0\\1\end{pmatrix}, Spec_\varepsilon(P_2)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},$$
$$Spec_\varepsilon(P_3)=\begin{pmatrix}1+\sqrt{3} & 1-\sqrt{3} & -2\\ 1 & 1 & 1\end{pmatrix}. \quad (8)$$

(ii) If $n=2k$ and $k\geq 2$, then

$$Spec_\varepsilon(P_{2k})=\begin{pmatrix}\frac{j+\sqrt{a}}{6} & \frac{-j+\sqrt{a}}{6} & 0 & \frac{j-\sqrt{a}}{6} & \frac{-j-\sqrt{a}}{6}\\ 1 & 1 & 2k-4 & 1 & 1\end{pmatrix}, \quad (9)$$

where $j=6k-3$ and $a=j(14k^2-20k+9)$.

(iii) If $n=2k+1$ and $k\geq 2$, then

$$Spec_\varepsilon(P_{2k+1})=\begin{pmatrix}\frac{6k+\sqrt{b}}{6} & \frac{-6k+\sqrt{c}}{6} & 0 & \frac{6k-\sqrt{b}}{6} & \frac{-6k-\sqrt{c}}{6}\\ 1 & 1 & 2k-3 & 1 & 1\end{pmatrix}, \quad (10)$$

where $b=6k(14k^2+3k+1)$ and $c=6k(14k^2-9k+1)$.

Proof:

(i) Proof of **Lemma 2** (i) already avaliable at the beginning of the proof of corresponding Lemma (namely **Lemma 2.1**) in **[11]**.

(ii) By the proof of **Lemma 2.1** (ii) in **[11]**, it obtained the eigenvalues of $\varepsilon(P_{2k})$ are $\frac{j+\sqrt{a}}{6}, \frac{j-\sqrt{a}}{6}, \frac{-j+\sqrt{a}}{6}$, and $\frac{-j-\sqrt{a}}{6}$, where $j = 6k-3$ and $a = j(14k^2-20k+9)$. The matrix $\varepsilon(P_{2k})$ also has $2k-4$ zero eigenvalues, where $k \geq 2$. According to **Definition 3**, the position of the eigenvalues on the ε-spectrum must be sorted from the largest to the smallest eigenvalue. For that reason, we investigated the connection of j and $\sqrt{a}$ when $k \geq 2$, in order to sort the eigenvalues on the ε-spectrum. The proof in **[11]** ignore this step and make **Lemma 1** (ii) does not hold.

Given $j = 6k-3$, $a = j(14k^2-20k+9)$, and $k \geq 2$. Claim $j > \sqrt{a}$, so it can be obtained that:

$$
\begin{aligned}
& j > \sqrt{a} \\
\Leftrightarrow\ & j^2 > a \\
\Leftrightarrow\ & j^2 - a > 0 \\
\Leftrightarrow\ & (6k-3)^2 - (6k-3)(14k^2-20k+9) > 0
\end{aligned}
$$

Using algebraic operations, the results of the inequality are $k < \frac{1}{2}$ or $\frac{6}{7} < k < 1$ which contradiction with the requirement $k \geq 2$. Consequently, $j < \sqrt{a}$, and because $k \geq 2$, j and $\sqrt{a}$ are always positive.

As $j < \sqrt{a}$ with j and $\sqrt{a}$ are always positive, then:

$(j+\sqrt{a}) > (-j+\sqrt{a}) > 0 > (j-\sqrt{a}) > (-j-\sqrt{a})$.

It makes $\frac{(j+\sqrt{a})}{6} > \frac{(-j+\sqrt{a})}{6} > 0 > \frac{(j-\sqrt{a})}{6} > \frac{(-j-\sqrt{a})}{6}$. Thus, the general form for $Spec_{\varepsilon}(P_{2k})$ when $k \geq 2$ as follows:

$$Spec_{\varepsilon}(P_{2k}) = \begin{pmatrix} \frac{j+\sqrt{a}}{6} & \frac{-j+\sqrt{a}}{6} & 0 & \frac{j-\sqrt{a}}{6} & \frac{-j-\sqrt{a}}{6} \\ 1 & 1 & 2k-4 & 1 & 1 \end{pmatrix}.$$

(iii) By the proof of **Lemma 2.1** (iii) in **[11]**, it obtained the eigenvalues of $\varepsilon(P_{2k+1})$ are $\frac{6k+\sqrt{b}}{6}, \frac{6k-\sqrt{b}}{6}, \frac{-6k+\sqrt{c}}{6}$, and $\frac{-6k-\sqrt{c}}{6}$, where $b = 6k(14k^2+3k+1)$, $c = 6k(14k^2-9k+1)$, and $k \geq 2$. The matrix $\varepsilon(P_{2k+1})$ also has $2k-3$ zero eigenvalues, where $k \geq 2$. According to **Definition 3**, the position of the eigenvalues on the ε-spectrum must be sorted from the largest to the smallest eigenvalue. For that reason, we investigated the connection of $\frac{6k+\sqrt{b}}{6}, \frac{6k-\sqrt{b}}{6}, \frac{-6k+\sqrt{c}}{6}, \frac{-6k-\sqrt{c}}{6}$, and 0, when $k \geq 2$, in order to sort the eigenvalues on the ε-spectrum. The proof in **[11]** ignore this step and make **Lemma 1** (iii) does not hold.

Recall that $k \geq 2$, $b = 6k(14k^2+3k+1)$, and $c = 6k(14k^2-9k+1)$. Since $6k$ and $\sqrt{b}$ are always positive, it holds $\frac{6k+\sqrt{b}}{6} > \frac{6k-\sqrt{b}}{6}$. As ε-eigenvalue is real, $\sqrt{c}$ is real positive. İt holds $\sqrt{c} > -6k$. Since $-6k$ is always negative and $\sqrt{c}$ is always positive, thus $\frac{-6k+\sqrt{c}}{6} > \frac{-6k-\sqrt{c}}{6}$. Because c has negative part, then $b > c$. Consequently, $\sqrt{b} > \sqrt{c}$.

Next, we check the connections of $6k$ with $\sqrt{b}$ and $6k$ with $\sqrt{c}$. At first, Claim $6k > \sqrt{b}$. Note that:

$$
\begin{aligned}
& 6k > \sqrt{b} \\
\Leftrightarrow\ & (6k)^2 > b \\
\Leftrightarrow\ & (6k)^2 - b > 0 \\
\Leftrightarrow\ & (6k)^2 - 6k(14k^2+3k+1) > 0 \\
\Leftrightarrow\ & 6k(6k-14k^2-3k-1) > 0 \\
\Leftrightarrow\ & 6k(-14k^2+3k-1) > 0
\end{aligned}
$$

Using algebraic operations is obtained that $k < 0$, which contradiction with $k \geq 2$. Consequently, $6k < \sqrt{b}$. In other words, $6k - \sqrt{b} < 0$

Given $c = 6k(14k^2-9k+1)$. Claim $6k > \sqrt{c}$, then it follows:

$$
\begin{aligned}
& 6k > \sqrt{c} \\
\Leftrightarrow\ & (6k)^2 > c
\end{aligned}
$$

$\Leftrightarrow (6k)^2 - c > 0$

$\Leftrightarrow (6k)^2 - 6k(14k^2 - 9k + 1) > 0$

$\Leftrightarrow 6k(6k - 14k^2 + 9k - 1) > 0$

$\Leftrightarrow 6k(-14k^2 + 15k - 1) > 0$

Using algebraic operations, the result of the inequality are $k < 0$ or $\frac{1}{14} < k < 1$, which contradiction with $k \geq 2$. So, it can conclude that $6k < \sqrt{c}$, or in other words $-6k + \sqrt{c} > 0$.

Since $6k - \sqrt{b} < 0$ and $-6k + \sqrt{c} > 0$, then $\frac{-6k+\sqrt{c}}{6} > \frac{6k-\sqrt{b}}{6}$. It makes $\frac{6k-\sqrt{b}}{6}$ and $\frac{-6k-\sqrt{c}}{6}$ are always negative, also makes both $\frac{6k+\sqrt{b}}{6}$ and $\frac{-6k+\sqrt{c}}{6}$ are positive.

In previous steps, we got $\frac{6k+\sqrt{b}}{6} > \frac{6k-\sqrt{b}}{6}$ and $\frac{-6k+\sqrt{c}}{6} > \frac{-6k-\sqrt{c}}{6}$.

As $\sqrt{b} > \sqrt{c}$, it holds $\frac{6k+\sqrt{b}}{6} > \frac{-6k+\sqrt{c}}{6} > 0 > \frac{6k-\sqrt{b}}{6} > \frac{-6k-\sqrt{c}}{6}$.

Thus, the general form for $Spec_\varepsilon(P_{2k+1})$ where $k \geq 2$ as follows:

$$Spec_\varepsilon(P_{2k+1}) = \begin{pmatrix} \frac{6k+\sqrt{b}}{6} & \frac{-6k+\sqrt{c}}{6} & 0 & \frac{6k-\sqrt{b}}{6} & \frac{-6k-\sqrt{c}}{6} \\ 1 & 1 & 2k-3 & 1 & 1 \end{pmatrix}.$$

Lemma 2 (ii) and (iii) are the corrections of **Lemma 1** (ii) and (iii). Based on **Example 5** (ii), we get the eigenvalues of $\varepsilon(P_8)$ and $Spec_\varepsilon(P_8)$. By using **Lemma 2** (ii) for $k = 4$, we obtain $j = 6(4) - 3 = 21$ and $a = 21(14(4)^2 - 20(4) + 9) = 3213$, so that:

$$Spec_\varepsilon(P_8) = \begin{pmatrix} \frac{21+\sqrt{3213}}{6} & \frac{-21+\sqrt{3213}}{6} & 0 & \frac{21-\sqrt{3213}}{6} & \frac{-21-\sqrt{3213}}{6} \\ 1 & 1 & 4 & 1 & 1 \end{pmatrix}$$
$$\approx \begin{pmatrix} 12.9472 & 5.9472 & 0 & -5.9472 & -12.9472 \\ 1 & 1 & 4 & 1 & 1 \end{pmatrix}.$$

In other hand, based on **Example 5** (iii), we also get $Spec_\varepsilon(P_9)$. By using **Lemma 2** (iii) when $k = 4$, we obtain $b = 6(4)(14(4^2) + 3(4) + 1) = 5688$ and $c = 6(4)(14(4^2) - 9(4) + 1) = 4536$, so the ε-spectrum of P_9 is:

$$Spec_\varepsilon(P_8) = \begin{pmatrix} \frac{24+\sqrt{5688}}{6} & \frac{-24+\sqrt{4536}}{6} & 0 & \frac{24-\sqrt{5688}}{6} & \frac{-24-\sqrt{4536}}{6} \\ 1 & 1 & 5 & 1 & 1 \end{pmatrix}$$
$$\approx \begin{pmatrix} 16.5698 & 7.2250 & 0 & -8.5698 & -15.2250 \\ 1 & 1 & 5 & 1 & 1 \end{pmatrix}.$$

The next discussion is related to the ε-energy of path graphs.

Theorem 1. [11] *Let P_n be a path with order n.*

(i) If $n = 1,2,3$ then

$$E_\varepsilon(P_1) = 0, E_\varepsilon(P_2) = 2, E_\varepsilon(P_3) = 2\sqrt{3} + 2. \tag{11}$$

(ii) If $n = 2k$ and $k \geq 2$, then

$$E_\varepsilon(P_{2k}) = \frac{2}{3}\sqrt{a}, \tag{12}$$

where $a = (6k - 3)(14k^2 - 20k + 9)$.

(iii) If $n = 2k + 1$ and $k \geq 2$, then

$$E_\varepsilon(P_{2k+1}) = \frac{1}{3}(\sqrt{b} + \sqrt{c}), \tag{13}$$

where $b = 6k(14k^2 + 3k + 1)$ and $c = 6k(14k^2 - 9k + 1)$.

Example 6. This example related to **Theorem 1** about ε-energy of paths. In previous example (**Example 5**), the ε-eigenvalues of P_1, P_2, P_3, P_8, and P_9 was calculated. Those ε-eigenvalues is used to calculate ε-energy of of P_1, P_2, P_3, P_8, and P_9 in this example, according to the definition of ε-energy in **Definition 4**.

(i) For $n = 1$, $n = 2$, and $n = 3$, we get graphs P_1, P_2, and P_3 respectively, as can be seen in **Figure 2.** Suppose A, B, C respectively are sets of the eigenvalues of $\varepsilon(P_1), \varepsilon(P_2)$, and $\varepsilon(P_3)$. According to the results in **Example 5** (i), $A = \{0\}, B = \{1, -1\}$, dan $C = \{1 + \sqrt{3}, 1 - \sqrt{3}, -2\}$. Using **Definition 4** and the concept of absolute value, it can be obtained:

$E_\varepsilon(P_1) = |0| = 0,$
$E_\varepsilon(P_2) = |1| + |-1| = 2,$
$E_\varepsilon(P_3) = |1 + \sqrt{3}| + |1 - \sqrt{3}| + |-2| = 1 + \sqrt{3} - 1 + \sqrt{3} + 2 = 2\sqrt{3} + 2.$

For n even, choose $k = 4$ so that $n = 2k = 8$. Graph P_8 illustrated in **Figure 3.** According to the results in **Example 5** (ii), ε-eigenvalues of P_8 are $-12.9472, -5.9472, 0, 0, 0, 0, 5.9472$, and 12.9472. By using **Definition 4**, the ε-energy of P_8:

$$E_\varepsilon(P_8) = |-12.9472| + |-5.9472| + 4|0| + |5.9472| + |12.9472| = 37.7888.$$

On the other hand, since $k = 4$ and $a = (6k - 3)(14k^2 - 20k + 9)$, then:

$$E_\varepsilon(P_8) = \frac{2}{3}\sqrt{a} = \frac{2}{3}\sqrt{3213} \approx \frac{2}{3} \times 56.6833 = 37.7888.$$

(ii) For n odd, choose $k = 4$, so as $n = 2k + 1 = 9$. Graph P_9 can be seen in **Figure 4.** Using the results of ε-eigenvalues of P_9 in **Example 5** (iii) and the definition of ε-energy in **Definition 4**, it can be obtained that:

$$E_\varepsilon(P_9) = |-15.2250| + |-8.5698| + 5|0| + |\ 7.2250| + |16.5698| = 47.5896.$$

On the other hand, since $k = 4$, $b = 6k(14k^2 + 3k + 1)$, and $c = 6k(14k^2 - 9k + 1)$ then:

$$E_\varepsilon(P_9) = \frac{1}{3}\left(\sqrt{b} + \sqrt{c}\right) = \frac{1}{3}\left(\sqrt{5688} + \sqrt{4536}\right) \approx \frac{1}{3} \times 142.7687 \approx 47.5896.$$

3.3 The ε-Spectrum and ε-Energy of Cycle Graphs

This subsection explains theorems and related examples of ε-eigenvalues, ε-spectrum, and ε-energy of cycle graphs. **Theorem 2** explains about the properties of ε-spectrum of even-order cycles and ε-eigenvalues of odd-order cycles. **Theorem 3** disscus about the properties of ε-energy of cycles with various order. Then, **Theorem 4** corrected an error in **Theorem 3** (ii).

Theorem 2. [4] *Let C_n be a cycle graph with order n.*

(i) If $n = 2k$, then

$$Spec_\varepsilon(C_{2k}) = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}. \quad (14)$$

(ii) If $n = 2k + 1$, then the ε-eigenvalues of C_{2k+1} are

$$\eta_i = 2k\cos\left(\frac{2\pi i}{2k + 1}\right); i = 1,2,3, \dots, 2k + 1. \quad (15)$$

Example 7. Choose $k = 4$. Thus, $n = 8$ for $n = 2k$ and $n = 9$ for $n = 2k + 1$. Graph C_8 and C_9 illustrated in **Figure 5.**

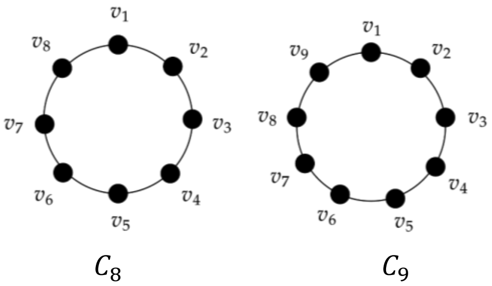

Figure 5. Graphs C_8 and C_9

(i) For graph C_8, using analogous steps on **Example 1** and **Example 2**, the eccentricity matrix obtained as follows:

$$\varepsilon(C_8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, the eigenvalues of $\varepsilon(P_8)$ are $-4, -4, -4, -4, 4, 4{,}4$ and 4. Thus, according to **Definition 3** and since $k = 4$, we can get $Spec_\varepsilon(P_8) = \begin{pmatrix} 4 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} k & -k \\ k & k \end{pmatrix}$.

(ii) For graph C_9, using analogous steps on **Example 1** and **Example 2**, it can be obtained that:

$$\varepsilon(C_9) = \begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As a results, the eigenvalues of $\varepsilon(P_9)$ with 4 decimal places are $-7.5175, -7.5175, -4, -4, 1.3892, 1.3892, 6.1284, 6.1284$, and 8. On the other hand, as $k = 4$, noted that we can get:

- $i = 1, \eta_1 = 8\,cos\left(\frac{2\pi}{9}\right) \approx 6.1284$
- $i = 2, \eta_2 = 8\,cos\left(\frac{4\pi}{9}\right) \approx 1.3892$
- $i = 3, \eta_3 = 8\,cos\left(\frac{6\pi}{9}\right) = -4$
- $i = 4, \eta_4 = 8\,cos\left(\frac{8\pi}{9}\right) \approx -7.5175$
- $i = 5, \eta_5 = 8\,cos\left(\frac{10\pi}{9}\right) \approx -7.5175$
- $i = 6, \eta_6 = 8\,cos\left(\frac{12\pi}{9}\right) = -4$
- $i = 7, \eta_7 = 8\,cos\left(\frac{14\pi}{9}\right) \approx 1.3892$
- $i = 8, \eta_8 = 8\,cos\left(\frac{16\pi}{9}\right) \approx 6.1284$
- $i = 9, \eta_9 = 8\,cos(2\pi) = 8$

As a result, eigenvalues of $\varepsilon(P_9)$ holds the form $\eta_i = 2k\,cos\left(\frac{2\pi i}{2k+1}\right)$ for $i = 1,2,3, \dots 9$.

The following theorem is related to the ε-energy of cycle graphs.

Theorem 3. [11] *Let C_n be a cycle graph with order n.*

(i) If $n = 2k$, then

$$E_\varepsilon(C_{2k}) = 2k^2. \tag{16}$$

(ii) If $n = 2k + 1$, then

$$E_\varepsilon(C_{2k+1}) = \frac{k}{sin\left(\frac{\pi}{2k+1}\right)}\left(sin\left(\frac{4k+3}{2k+1}\pi\right) - sin\left(\frac{1}{2k+1}\pi\right)\right). \tag{17}$$

Example 8. This example related to **Theorem 3** about ε-energy of cycles. In previous example (**Example 7**), the ε-eigenvalues of C_8 and C_9 was calculated. Those ε-eigenvalues is used in this examples to calculate ε-energy of of C_8 and C_9, according to the definition of ε-energy in **Definition 4**.

(i) For n even, in **Example 7** (i) has been obtained the ε-*eigenvalues* of C_8 are $-4, -4, -4, -4, 4, 4, 4,$ dan 4. According to the definition of ε-energy in **Definition 4** and since $k = 4$, we get:

$$E_\varepsilon(C_8) = 4|-4| + 4|4| = 4^2 + 4^2 = 2(4^2) = 2k^2.$$

(ii) For n odd, in **Example 7** (ii) has been obtained the ε-*eigenvalues* of C_9 are $-7.5175, -7.5175, -4, -4, 1.3892, 1.3892, 6.1284, 6.1284,$ and 8. According to the definition of ε-energy in **Definition 4,** we can get:

$$E_\varepsilon(C_9) = 2|-7.5175| + 2|-4| + 2|1.3892| + 2|6.1284| = 32.4702.$$

On the other hand, since $k = 4$, the value of:

$$\frac{k}{\sin\left(\frac{\pi}{2k+1}\right)}\left(\sin\left(\frac{4k+3}{2k+1}\pi\right) - \sin\left(\frac{1}{2k+1}\pi\right)\right) = \frac{4}{\sin\left(\frac{\pi}{9}\right)}\left(\sin\left(\frac{19}{9}\pi\right) - \sin\left(\frac{1}{9}\pi\right)\right) = 0 \neq 32.4702.$$

Consequently, **Theorem 3** (ii) does not hold.

Theorem 3 (ii) does not hold in counter example in **Example 8** (ii) because there is a small error on the proof of corresponding Theorem (namely **Theorem 2.2**) in **[11]**. This error is caused by the proof in **[11]** not considering whether the value of $\cos\left(\frac{2\pi i}{2k+1}\right)$ is positive or negative. In fact, ε-energy is the absolute sum of ε-eigenvalues. So, on the beginning of the proof, it should be considered the use of the absolute value concept according to the definition of ε-energy as in **Definition 4**. The general correction of **Theorem 3** can be written as **Theorem 4** below:

Theorem 4. *Let C_n be a cycle graph with order n.*

(i) If $n = 2k$, then

$$E_\varepsilon(C_{2k}) = 2k^2. \tag{18}$$

(ii) If $n = 2k + 1$, then

$$E_\varepsilon(C_{2k+1}) = \sum_{i=1}^{2k+1}\left|2k\cos\left(\frac{2\pi i}{2k+1}\right)\right|. \tag{19}$$

Proof.

(i) Proof of **Theorem 4** (i) has already available at the beginning of the proof of corresponding Theorem (namely **Theorem 2.2**) in **[11]**.

(ii) For any graph C_{2k+1}, since ε-energy is the absolute sum of ε-eigenvalues, the general form $E_\varepsilon(C_{2k+1})$ can be found using **Theorem 2** (ii) and the concept of absolute value. Based on **Theorem 2** (ii), the ε-eigenvalues of graph C_{2k+1} satisfy:

$$\eta_i = 2k\cos\left(\frac{2\pi i}{2k+1}\right); i = 1,2,3, \dots, 2k+1.$$

Therefore, to find the general formula of ε-energy of cycle with odd-order, its need to look for the patterns of $\sum_{i=1}^{n}\left|2k\cos\left(\frac{2\pi}{n}i\right)\right|$ for n odd and $n \geq 1$, if its possible. In other words, according to definition of ε-energy as in **Definition 4** and the properties of ε-eigenvalues of graph C_{2k+1} in **Theorem 2** (ii), it can simply write as:

$$E_\varepsilon(C_{2k+1}) = \sum_{i=1}^{2k+1}\left|2k\cos\left(\frac{2\pi i}{2k+1}\right)\right|,$$

without finding its pattern. So, **Theorem 4** (ii) proved.

4. CONCLUSIONS

Based on the studies that have been conducted in this article, the following conclusions can be drawn:

i) Both of eccentricity spectrum and eccentricity energy are representations of the eigenvalues of eccentricity matrix.

ii) Eccentricity spectrum and eccentricity energy of path that have 1, 2, and 3 vertices have a special form. Meanwhile, eccentricity spectrum and eccentricity energy of path with even and odd order have different formulas. The correction in lemma related to eccentricity spectrum of path graphs with order more than 3 is obtained on **Lemma 2**.

iii) Eccentricity spectrum and eccentricity energy of odd order cycles have different properties than even order cycles. The correction in theorem related to eccentricity energy of cycle graphs with odd-order are obtained on **Theorem 4**.

Future research can study the eccentricity spectrum and eccentricity energy in graph classes other than paths and cycles. Besides that, the future research can propose the concept of eccentricity matrix and its properties on other types of graphs, for example fuzzy graphs.

REFERENCES

[1] F. Arary, *Graph Theory*. New York: CRC Press, 2018.

[2] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*, 6th ed. Boca Raton: CRC Press, 2016.

[3] J. A. Bondy and U. S. R. Murty, *Graduate Texts in Mathematics Series: Graph Theory*, 244th ed. USA: Springer, 2008.

[4] J. Wang, M. Lu, F. Belardo, and M. Randić, "The anti-adjacency matrix of a graph: Eccentricity matrix," *Discret. Appl. Math.*, vol. 251, pp. 299–309, 2018.

[5] M. Randić, "DMAX – Matrix of Dominant Distances in a Graph," *MATCH Commun. Math. Comput. Chem.*, vol. 70, pp. 221–238, 2013.

[6] R. Stin, S. Aminah, and S. Utama, "Characteristic polynomial and eigenvalues of the anti- adjacency matrix of cyclic directed prism graph," in *Proceedings of the 4th International Symposium on Current Progress in Mathematics and Sciences*, 2019.

[7] M. I. A. Prayitno, S. Utama, and S. Aminah, "Properties of anti-adjacency matrix of directed cyclic sun graph," in *IOP Conf. Series: Materials Science and Engineering*, 2019, p. 012020.

[8] I. Jeyaraman and T. Divyadevi, "On Eccentricity Matrices of Wheel Graphs," *arXiv*, vol. 1, 2020.

[9] I. Mahato and M. R. Kannan, "On the eccentricity matrices of trees: Inertia and spectral symmetry," *Discrete Math.*, vol. 345, p. 113067, 2022.

[10] X. Yang and L. Wang, "The eccentricity matrix of a digraph," *Discret. Appl. Math.*, vol. 322, pp. 61–73, 2022.

[11] J. Wang, L. Lu, M. Randić, and G. Li, "Graph energy based on the eccentricity matrix," *Discrete Math.*, vol. 342, pp. 2636–2646, 2019.

[12] F. Tura, "On the eccentricity energy of complete multipartite graph," *arXiv*, vol. 1, 2020.

[13] S. S. Khunti, J. A. Gadhiya, M. A. Chaurasiya, and M. P. Rupani, "Eccentricity energy of bistar graph and some of its related graphs," *Malaya J. Mat.*, vol. 8, no. 4, pp. 1464–1468, 2020.

[14] W. Wei, S. Li, and L. Zhang, "Characterizing the extremal graphs with respect to the eccentricity spectral radius, and beyond," *Discrete Math.*, vol. 345, p. 112686, 2022.

[15] J. Li, L. Qiu, and J. Zhang, "Proof of a conjecture on the -spectral radius of trees," *AIMS Math.*, vol. 8, no. 2, pp. 4363–4371, 2022.

[16] W. Wei and S. Li, "On the eccentricity spectra of complete multipartite graphs," *Appl. Math. Comput.*, vol. 424, p. 127036, 2022.

[17] Z. Qiu, Z. Tang, and Q. Li, "Eccentricity spectral radius of t-clique trees with given," *Discret. Appl. Math.*, vol. 337, pp. 202–217, 2023.