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CONSTRUCTION OF BICYCLIC GRAPH AND ITS APPLICATION IN TRANS JOGJA ROUTES

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ABSTRACT

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A bicyclic graph is a type of graph that consists of exactly two cycles. A cycle is a graph that is a closed path where no vertices are repeated except the first and last vertices which are the same. The cycles in bicyclic graph can be of different lengths and shapes, but they must have at least one common vertex. Bicyclic graphs can be divided into two categories based on the types of induced subgraphs they contain. One category consists of graphs that include an ∞-graph as an induced subgraph, while the other category comprises graphs that contain a θ-graph as an induced subgraph. There are 3 types of bicyclic graph without pendant vertex. A directed graph, also referred to as a digraph, is a graph in which each edge is assigned a specific direction. A directed bicyclic graph is a special kind of directed graph that contains precisely two distinct directed cycles. This graph can be applied in transportation problem. In this article, we give some examples of directed bicyclic graph in Trans Jogja routes.

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1. INTRODUCTION

A graph can be defined as a combination of two elements: a non-empty set of finite vertices $V(G)$, and a set of edges $E(G)$, where each edge is a 2-element subset of $V(G)$. Vertices represent individual entities or elements, while edges represent the connections or relationships between the vertices. A pendant vertex is a specific type of vertex in a graph. It is defined as a vertex that has a degree of 1, meaning it is connected to only one other vertex through a single edge. A path is a type of simple graph where its vertices can be arranged in a linear sequence. When we have a path denoted as $P = x_0 \dots x_{k-1}$, with $k \geq 3$, we can create a cycle by adding an edge between the last vertex x_{k-1} and the first vertex x_0 . This resulting graph is called a cycle and is denoted as $C := P + x_{k-1}x_0$. A cycle of length k is specifically called a k-cycle and is represented as C_k. Theory of graph can be seen on [1, 2, 3, 4].

In graph theory, if we have a subgraph G' that is contained within a larger graph G, we write it as $G' \subseteq G$. When G' includes all the edges $xy \in E$ where both x and y are vertices in V', we consider G' to be an induced subgraph of G. In other words, the set of vertices V' in G induces or spans the subgraph G', and we can write this relationship as $G' = G[V']$ [1]. Two or more paths are considered independent when none of them share any intermediary vertices. In other words, if there are two paths from vertex a to vertex b, they are independent if the only vertices they have in common are a and b. A graph G is considered connected if it is not empty and there is a path linking any two vertices within G. If a subset U of vertex set in G, denoted as $U \subseteq V(G)$, forms a connected subgraph $G(U)$, we also refer to U as connected within G. Research on induced subgraphs can be observed in various studies and academic investigations. The concept of induced subgraphs has been a subject of interest in research, and scholars have explored its properties [5], applications in computation [6], and implication in coloring [7].

A bicyclic graph is a type of connected graph that consists two independent cycles [8]. This type of graph is denoted by B_n, representing the set of all bicyclic graphs of order n. In 2008, Hu et al. explained that there are two types of bicyclic graphs: the ∞-graph and the θ-graph. The ∞-graph, denoted as $\infty(p, l, q)$, is formed by combining two vertex-disjoint cycles, C_p and C_q. It includes a path, P_l, of length $l-1$ that connects a vertex v_i from C_p to a vertex v_l from C_q. It is important to note that p and q must be greater than or equal to 3, and l must be greater than or equal to 1. If l equals 1, it means that v_i is equal to v_l. The θ-graph, denoted as $\theta(p, l, q)$. Consider three vertex-disjoint paths, denoted as P_{p+1}, P_{l+1} and P_{q+1}, where the minimum of p, l, and q is at least 1, and at most one of them is 1. By identifying the initial and terminal vertices of these paths, the resulting graph is known as a θ-graph [9]. A θ-graph can be obtained by combining two cycles, which have k common vertices. [10].

In 2016, Ma et al. divided bicyclic graph without pendant vertex in three types. Consider a bicyclic graph with order n denoted as $\mathcal{B}_n$. Let $C_p = v_1 v_2 \dots v_p v_1$ and $C_q = u_1 u_2 \dots u_q u_1$ be two cycles in $\mathcal{B}$ with $l\ (l \geq 0)$ common vertices. If $l = 0$, there exists a unique path P that connects C_p and C_q, starting with v_1 and ending with u_1. This type of bicyclic graph is referred to as Type I $\left(\mathcal{B}_n^1(p, q)\right)$. If $l = 1, C_p$ and C_q have exactly one common vertex, $v_1 = u_1$. These types of bicyclic graphs are called Type II $\left(\mathcal{B}_n^2(p, q)\right)$. If $l \geq 2$, the two cycles C_p and C_q have more than one common vertex, $v_1 = u_1, \dots, v_l = u_l$.. These bicyclic graphs are referred to as Type III $\left(\mathcal{B}_n^3(p, l, q)\right)$ [11]. Based on Ma and Hu's explanation of the types of bicyclic graphs. Therefore, we review some construction of bicyclic graph and give some examples.

Moreover, a directed graph, also known as a digraph, is a type of graph that consists of a finite set of vertices and a subset of edges that have direction. In this paper, we follow to the theoretical terminologies related to directed graphs (digraphs) as presented in relevant books [12, 13, 14]. In 2023, Vo et al. constructed a research on the application of directed graphs in the field of transportation. Directed graphs were used to determine optimal transport [15].

Furthermore, a directed bicyclic graph is a bicyclic graph that has directed edges [16, 17]. The application of directed bicyclic graphs extend across diverse fields such as systems modeling [18], control theory [19], network analysis [20, 21], and transportation planning [22]. Based on the application of directed bicyclic graphs in transportation, we review the application of directed bicyclic graphs in the construction of the Trans Jogja route. Trans Jogja has 17 routes that spread across in DIY (Daerah Istimewa Yogyakarta) [23]. We review that the construction of directed bicyclic graphs lies in the Trans Jogja routes that connect one stop to another without direct routes. In this paper, we show application of directed bicyclic graph on Trans Jogja routes.

2. RESEARCH METHODS

This research focuses on studying the construction of bicyclic graphs and providing an example of their application in transportation, particularly in Trans Jogja routes. The construction of a bicyclic graph involves providing the concept of vertices and edges in the bicyclic graph. This construction is based on the types of bicyclic graphs. In addition to the construction, illustrations of each type of bicyclic graph are also provided. Bicyclic graphs are used as representations of combined, non-connected routes. By utilizing bicyclic graphs, alternative routes can be found that connect these separate routes. In this study, it is important to consider the directions of each route involved, as the direction will affect the distance that needs to be travel within those routes. This research will identify the construction of bicyclic graph in the Trans Jogja routes.

In the results and discussion section, the discussion is divided into several subsections. The first subsection discusses the types of bicyclic graphs, and provide some explanations and examples. The second subsection explains the development of bicyclic graph types that do not have pendant vertices. This subsection further categorizes the bicyclic graphs into three types and give an overview. Conclusions regarding the different types of bicyclic graphs are drawn. In the last subsection, the application of bicyclic graphs in the Trans Jogja routes is discussed. Specifically, the application focuses on directed bicyclic graphs.

3. RESULTS AND DISCUSSION

This section discusses the results and discussions of the research, which contain definition of bicyclic graph, types of bicyclic graphs, and application of directed bicyclic graphs in the Trans Jogja routes.

3.1 Bicyclic graph

In this subsection, we discuss the classifications of bicyclic graphs and explanations along with examples to illustrate each type. We give literature review of general definition of bicyclic graphs in **Definition 1**. Bicyclic graphs are classified into two types: ∞-graph and $\boldsymbol{\theta}$-graph. The definitions of ∞-graph and $\boldsymbol{\theta}$-graph are presented in **Definition 2** and **Definition 3**. The examples of ∞-graph and $\boldsymbol{\theta}$-graph are presented in **Example 2** and **Example 3**.

Definition 1. [9] A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one denoted by $\mathcal{B}_n$ the set of all bicyclic graphs of order n.

According to Hu et al. [9], the types of bicyclic graphs are as follows: ∞-graph and θ-graph.

Definition 2. [9] Let C_p and C_q be two vertex-disjoint cycles. ∞-graph $\left(\infty(p,l,q)\right)$ is obtained from two vertex-disjoint $v_i \in C_p$ and $v_l \in C_q$ with a path P_l of length $l-1$, where $p,q \geq 3$ and $l \geq 1$, $l=1$ means $v_i = v_l$.

The ∞-graph can be illustrated on **Figure 1. ∞-graph$\left(\infty(\boldsymbol{p},\boldsymbol{l},\boldsymbol{q})\right)$ Figure 1**.

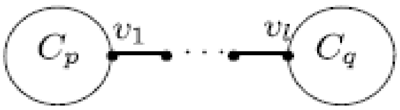


Figure 1. ∞-graph$\left(\infty(\boldsymbol{p},\boldsymbol{l},\boldsymbol{q})\right)$

Let $\mathcal{B}_n = \left(\infty(p,l,q)\right)$, then the set of vertices and edges is obtained as follows.

$$V(C_p) = \{x_i | i = 1,2,3,\dots,p\}$$
$$E(C_p) = \{x_i x_{i+1},\ x_p x_1 | i = 1,2,3,\dots,p-1\}$$
$$V(C_q) = \{u_j | j = 1,2,3,\dots,q\}$$
$$E(C_q) = \{u_i u_{i+1},\ u_q u_1 | i = 1,2,3,\dots,q-1\}$$

If $l > 1$, then

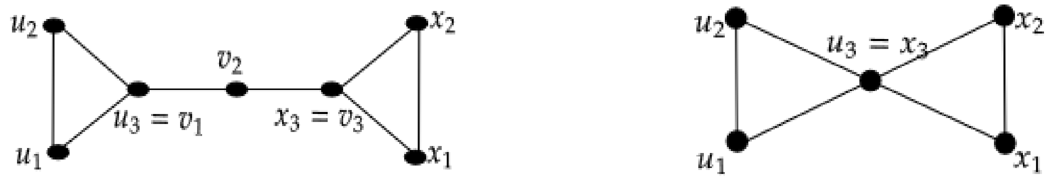
$V(\mathcal{B}_n) = \{x_i, u_j, v_k | i = 1,2,3,\dots,p; j = 1,2,3,\dots,q; k = 1,2,3,\dots,l;\ v_1 = x_i; v_l = u_j,\ l > 1\}$, otherwise

$V(\mathcal{B}_n) = \{x_i, u_j\ | i = 1,2,3,\dots,p; j = 1,2,3,\dots,q; x_i = u_j,\ l = 1\}$.

If $l > 1$, then

$E(\mathcal{B}_n) = E(C_P) \cup E(C_q) \cup \{v_k, v_{k+1} | k = 1,2,3, \dots, l-1; l > 1\}$, otherwise $E(\mathcal{B}_n) = E(C_P) \cup E(C_q)$; $l = 1$.

Example 1. For illustration, we give some examples of ∞-graph. Let $p = 3$ dan $q = 3$, then the set of vertices and edges are $V(C_p) = \{x_1, x_2, x_3\}, E(C_p) = \{x_1x_2, x_2x_3, x_3x_1\}$ and $V(C_q) = \{u_1, u_2, u_3\}, E(C_q) = \{u_1u_2, u_2u_3, u_3u_1\}$. We illustrate ∞- graph with $l > 1$ and $l = 1$ in Figure 2.



(a) (b)

Figure 2. Example of ∞-graph, (a) $\infty(3,3,3)$, (b) $\infty(3,1,3)$

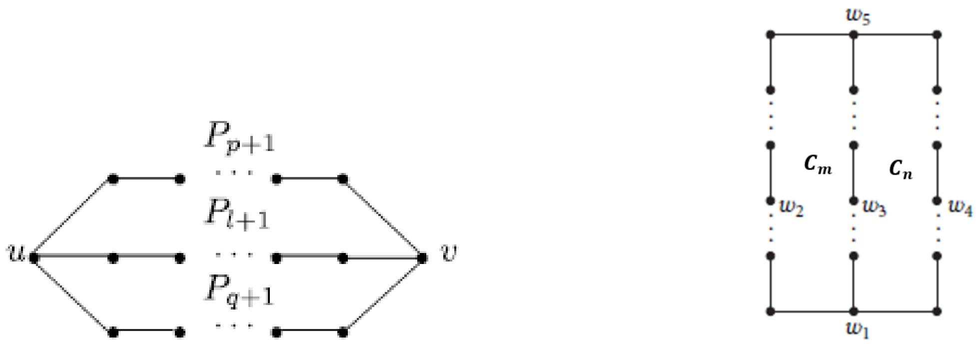
Figure 2 (a) Illustrates ∞-graph with $l = 3$. Let $\mathcal{B}_7 = (\infty(3,3,3))$, then the sets of vertices and edges are $V(\mathcal{B}_7) = \{x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3 | x_3 = v_3; u_3 = v_1\}$ and

$$E(\mathcal{B}_7) = \{x_1x_2, x_2x_3, x_3x_1, v_1v_2, v_2v_3, v_3v_1, u_1u_2, u_2u_3, u_3u_1 | x_3 = v_3; u_3 = v_1\}.$$

Figure 2 (b) Iillustrates ∞-graph with $l = 1$. Let $\mathcal{B}_5 = (\infty(3,1,3))$, then the sets of vertices and edges are $V(\mathcal{B}_5) = \{x_1, x_2, x_3, u_1, u_2, u_3 | x_3 = u_3\}$ and $E(\mathcal{B}_5) = \{x_1x_2, x_2x_3, x_3x_1, u_1u_2, u_2u_3, u_3u_1 | x_3 = u_3\}$.

Definition 3. [9] Let P_{p+1}, P_{l+1} and P_{q+1} be three vertex-disjoint paths, where $min(p, l, q) \geq 1$ and the minimum among p, l, and q is at least 1, and only one of them is allowed to be equal to 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resultant graph, denoted by $\theta(p, l, q)$, is called a θ-graph.

The θ-graph can be formed by three paths, namely P_{p+1}, P_{l+1}, and P_{q+1}. The θ-graph can also be formed by two cycles, C_m and C_n, which have k common vertices. The ilustration of θ-graph can be seen in Figure 3.



(a) (b)

Figure 3. Ilustration of θ-graph, (a) θ-graph formed by three paths, (b) θ-graph formed by two cycles

The bicyclic graph $\mathcal{B}_n = \theta(p, l, q)$ is obtained from three vertex-disjoint paths P_{p+1}, P_{l+1} dan P_{q+1} that have same initial and terminal vertices. The set of vertices and edges is obtained as follows.

$$V(P_{p+1}) = \{u_1, u_2, \dots, u_p, u_{p+1}\}$$

$$E(P_{p+1}) = \{u_1u_2, u_2u_3, \dots, u_{p-1}u_p, u_pu_{p+1}\}$$

$$V(P_{l+1}) = \{v_1, v_2, \dots, v_l, v_{l+1}\}$$

$$E(P_{l+1}) = \{v_1v_2, v_2v_3, \dots, v_{l-1}v_l, v_lv_{l+1}\}$$

$$V(P_{q+1}) = \{w_1, w_2, \dots, w_q, w_{q+1}\}$$

$$E(P_{l+1}) = \{w_1w_2, w_2w_3, \dots, w_{q-1}w_q, w_qw_{q+1}\}$$

$$V(\mathcal{B}_n) = \{V(P_{p+1}) \cup V(P_{l+1}) \cup V(P_{q+1}) | u_1 = v_1 = w_1, u_{p+1} = v_{l+1} = w_{p+1}\}$$

$$E(\mathcal{B}_n) = \{E(P_{p+1}) \cup E(P_{l+1}) \cup E(P_{q+1}) | u_1 = v_1 = w_1, u_{p+1} = v_{l+1} = w_{p+1}\}$$

Example 2. For illustration, we give an example of θ-graph. Let there are 3 path, P_{4+1}, P_{3+1} and P_{4+1}, then the set of vertices and edges are $V(P_{4+1}) = \{u_1, u_2, u_3, u_4, u_5\}, E(C_p) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5\}$ and $V(P_{3+1}) = \{v_1, v_2, v_3 v_4\}, E(P_{3+1}) = \{v_1v_2, v_2v_3, v_3v_4\}$ and $V(P_{4+1}) = \{w_1, w_2, w_3, w_4, w_5\}$, $E(C_p) = \{w_1w_2, w_2w_3, w_3w_4, w_4w_5\}$. We illustrate θ- graph with 10 vertices in **Figure 4.**

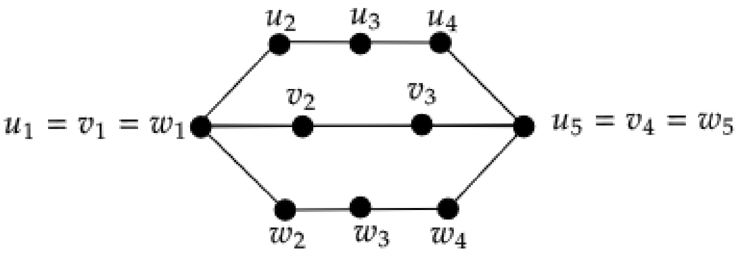


Figure 4. An example of θ-graph $\mathcal{B}_{10} = (\theta(4,3,3))$

Figure 4 illustrate an θ-graph with 10 vertices denoted by $\mathcal{B}_{10} = (\theta(4,3,3))$. The sets of vertices and edges are $V(\mathcal{B}_{10}) = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3 v_4, w_1, w_2, w_3, w_4, w_5 | u_1 = v_1 = w_1; u_5 = v_4 = w_5\}$ and $E(\mathcal{B}_{10}) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, v_1v_2, v_2v_3, v_3v_4, w_1w_2, w_2w_3, w_3w_4, w_4w_5 | u_1 = v_1 = w_1; u_5 = v_4 = w_5\}$.

Bicyclic graphs can be described based on the number of vertices. Let $\widehat{\mathcal{B}}_n$ is a set containing θ-graphs dan ∞-graphs with n vertices. Here, we illustrated some examples of the sets of all bicyclic graphs of order $n; n = 4,5,6$.

Example 3. If $n = 4$, then $\mathcal{B}_4 = \widehat{\mathcal{B}}_4 = \theta(2,1,2)$. The ilustration of $\mathcal{B}_4$ can be seen on **Figure 5.**

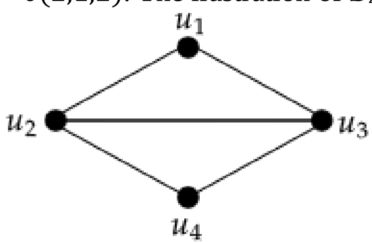


Figure 5. An example of bicyclic graph with 4 vertices $\widehat{\mathcal{B}}_4 = \theta(2,1,2)$

Example 4. If $n = 5$, then $\widehat{\mathcal{B}}_5 = (\theta(2,2,2), \theta(2,1,3), \infty(3,1,3)), \widehat{\mathcal{B}}_4 + P_1$ where P_1 incident with pendent vertex. (see on **Figure 6)**

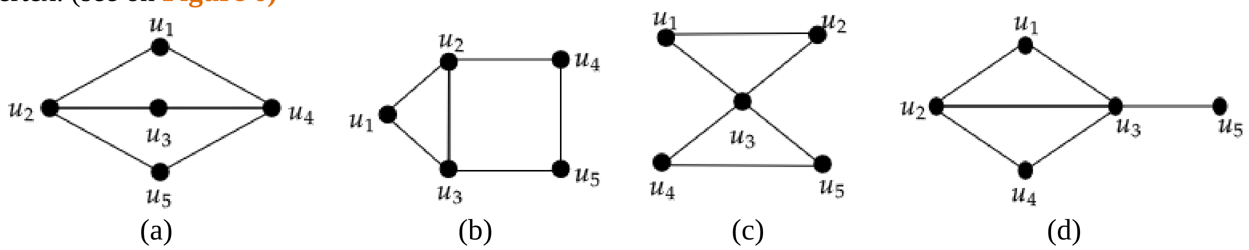
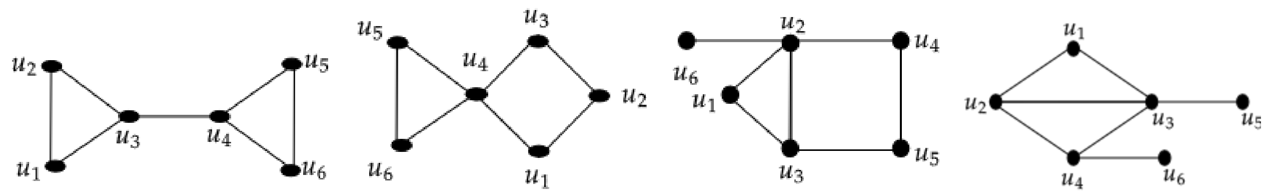


Figure 6. Some examples of bicyclic graphs with 5 vertices $\widehat{\mathcal{B}}_5$, (a) $\theta(2,2,2)$, (b) $\theta(2,1,3)$, (c) $\infty(3,1,3)$ and (d) $\widehat{\mathcal{B}}_4 + P_1 = \theta(2,1,2) + P_1$

Example 5. If $n = 6$, then $\mathcal{B}_6 = \widehat{\mathcal{B}}_6(\theta(2,3,2), \theta(2,1,4), \theta(3,1,3), \infty(3,1,4), \infty(3,2,3)), \mathcal{B}_5 + P_1, \widehat{\mathcal{B}}_4 + 2P_1$ where P_1 incident with pendent vertex. **Figure 7** illutrates an **Example 5.**



(a) (b) (c) (d)

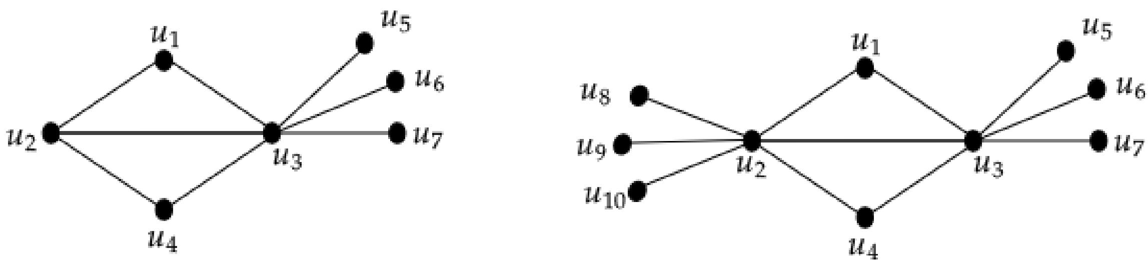
Figure 7. Some Examples of Bicyclic Graph With 6 Vertices $\mathcal{B}_6$, (a) $\infty(3,2,3)$, (b) $\infty(3,1,4)$, (c) $\widehat{\mathcal{B}}_5 + P_1$ and (d) $\widehat{\mathcal{B}}_4 + 2P_1$

Bicyclic graph of order n with s pendant vertices can be denoted by $\mathcal{B}_n(s)$ where $s \geq 1$. It contains s pendant vertices, which are isolated vertices connected to only one vertex in the graph. These pendant vertices do not participate in the cycles or the connecting path. The presence of the cycles, path, and pendant vertices characterizes the structure of a bicyclic graph of order n with s pendant vertices. Bicyclic graph of order n with s and t pendant vertex could denote by $\mathcal{B}_n(s,t)$ where $s,t \geq 1$, It also includes s pendant vertices connected to one cycle and t pendant vertices connected to the other cycle. The pendant vertices are isolated vertices that are linked to only one vertex in the graph and do not participate in the cycles or the connecting path. The structure of a bicyclic graph of order n with s and t pendant vertices is determined by the presence of the cycles, path, and pendant vertices. Bicyclic graph with pendant vertex illustrated in Figure 8.

(a) (b)

Figure 8. Bicyclic graph with pendant vertices, (a) $\mathcal{B}_n(s); n = s + 5$, (b) $\mathcal{B}_n(s,t); n = s + t + 5$

Example 6. For illustration, we give an example of bicyclic graph that have pendant vertex. Figure 9 illustrate example of $\mathcal{B}_n(s)$ and $\mathcal{B}_n(s,t)$, that are $\mathcal{B}_7(3)$ and $\mathcal{B}_{10}(3{,}3)$.



(a) (b)

Figure 9. Example of bicyclic graph with pendant vertices, (a) $\mathcal{B}_7(3)$, (b) $\mathcal{B}_{10}(3,3)$

3.2 Types of bicyclic graph

In this subsection, we explained about bicyclic graph types that do not contain pendant vertices. Bicyclic graphs without pendant vertices are classified into three types. The definition and examples of each type of bicyclic graph are described.

Definition 4. [11] Let $\mathcal{B}$ be a bicyclic graph. Suppose $C_p = v_1 v_2 \dots v_p v_1$ and $C_q = u_1 u_2 \dots u_q u_1$ are two cycles in $\mathcal{B}$ with $l (l \geq 0)$ common vertices.

(i) If $l = 0$, then there is one unique path P connecting Cp and Cq, which initiates with v_1 and concludes with u_1. We call this kind of bicyclic graph type I, and it is denoted by $\mathcal{B}_n^1(p,q)$.

(ii) If $l = 1$, then C_p and C_q have exactly one common vertex $v_1 = u_1$. We call this kind of bicyclic graphs type II, and it is denoted by $\mathcal{B}_n^2(p,q)$.

(iii) If $l \geq 2$, then the two cycles C_p and C_q have more than one common vertex $v_1 = u_1, \dots, v_l = u_l$. We call this kind of bicyclic graphs type III, and it is denoted by $\mathcal{B}_n^3(p,l,q)$.

The illustation of the three types of bicyclic graphs can be seen in **Figure 10**.

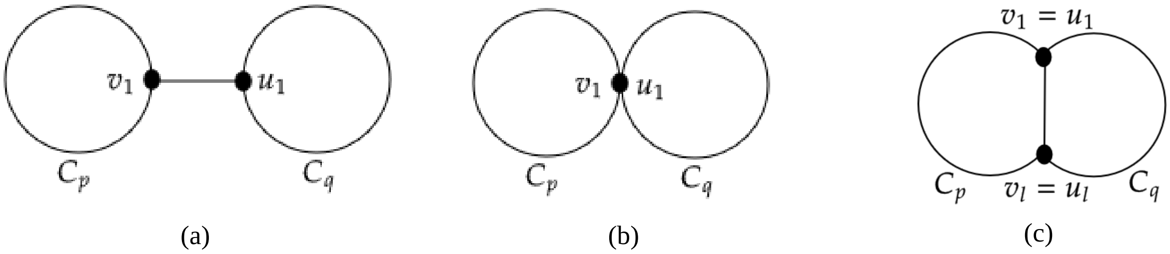


Figure 10. Three types of bicyclic graphs, (a) bicyclic graph type I, (b) bicyclic graph type II, (c) bicyclic graph type III

Next, we conclude the construction of the bicyclic graph. The set of all bicyclic graphs of order n, denoted by $\mathcal{B}_n$ where $n \geq 4$, is given by$\mathcal{B}_n = \mathcal{B}_n^1 \cup \mathcal{B}_n^2 \cup \mathcal{B}_n^3$.

For $\mathcal{B}_n^1(p,q)$ with one unique path P_k where $k = n - p - q + 2$, the set of vertices and egdes is obtained as follows.

$$V(C_p) = \{v_1, v_2, \dots, v_p\}$$
$$E(C_p) = \{v_1v_2, v_2v_3, \dots, v_{p-1}v_p, v_pv_1\}$$
$$V(C_q) = \{u_1, u_2, \dots, u_q\}$$
$$E(C_q) = \{u_1u_2, u_2u_3, \dots, u_{q-1}u_q, u_qu_1\}$$
$$V(\mathcal{B}_n^1) = V(C_p) \cup V(C_q) \cup \{x_i | i = 1,2,3, \dots, k; x_1 = v_1; x_k = u_k\}$$
$$E(\mathcal{B}_n^1) = E(C_p) \cup E(C_q) \cup \{x_ix_{i+1} | i = 1,2,3, \dots, k-1; x_1 = v_1; x_k = u_k\}$$

Example 7. Bicyclic graph Type I with 8 vertices, where $p = 3$ and $q = 4$ denoted $\mathcal{B}_8^1(3,4)$. The sets of vertices and edges of cyclic graph are $V(C_p) = \{v_1, v_2, v_3\}, E(C_p) = \{v_1v_2, v_2v_3, v_3v_1\}$ and $V(C_q) = \{u_1, u_2, u_3, u_4\}, E(C_q) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$. This example is depicted by **Figure 11.**

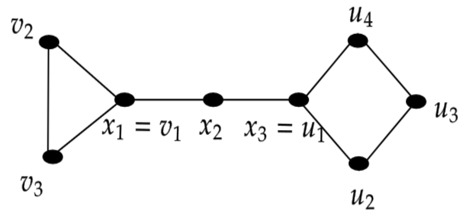


Figure 11. An example of bicyclic graph type I $\mathcal{B}_8^1(3,4)$

Figure 11 illustrates $\mathcal{B}_8^1(3,4)$. The sets of vertices and edges are

$V\left(\mathcal{B}_8^1(3,4)\right) = \{v_1, v_2, v_3, u_1, u_2, u_3, u_4, x_1, x_2, x_3 | x_1 = v_1; x_3 = u_1\}$ and

$E\left(\mathcal{B}_8^1(3,4)\right) = \{v_1v_2, v_2v_3, v_3v_1, x_1x_2, x_2x_3, u_1u_2, u_2u_3, u_3u_4, u_4u_1 | x_1 = v_1; x_3 = u_1\}$

Let $\mathcal{B}_n^2(p,q)$, then the set of vertices and egdes is obtained as follows.

$$V(C_p) = \{v_1, v_2, \dots, v_p\}$$
$$E(C_p) = \{v_1v_2, v_2v_3, \dots, v_{p-1}v_p, v_pv_1\}$$
$$V(C_q) = \{u_1, u_2, \dots, u_q\}$$
$$E(C_q) = \{u_1u_2, u_2u_3, \dots, u_{q-1}u_q, u_qu_1\}$$
$$V(\mathcal{B}_n^2) = V(C_p) \cup V(C_q); v_1 = u_1$$
$$E(\mathcal{B}_n^2) = E(C_p) \cup E(C_q); v_1 = u_1$$

Example 8. In **Figure 12**, an example of bicyclic graph type II is depicted, denoted as $\mathcal{B}_7^2(4,4)$, where

$p = 4$ and $q = 4$. The sets of vertices and edges of cyclic graph are $V(C_p) = \{v_1, v_2, v_3, v_4\}$, $E(C_p) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ and $V(C_q) = \{u_1, u_2, u_3, u_4\}, E(C_q) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$.

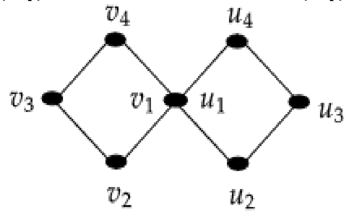


Figure 12. An example of bicyclic graph type II $\mathcal{B}_7^2(4,4)$

Illustrating $\mathcal{B}_7^2(4,4)$ in **Figure 12**, the sets of vertices and edges are given by:

$V(\mathcal{B}_7^2(4,4)) = \{v_1, v_2, v_3, u_1, u_2, u_3, u_4, | v_1 = u_1\}$ and

$E(\mathcal{B}_7^2(4,4)) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, , u_1u_2, u_2u_3, u_3u_4, u_4u_1 | v_1 = u_1\}$.

Let $\mathcal{B}_n^3(p, l, q)$, then the set of vertices and egdes is obtained as follows.

$$V(C_p) = \{v_1, v_2, \dots, v_p\}$$
$$E(C_p) = \{v_1v_2, v_2v_3, \dots, v_{p-1}v_p, v_pv_1\}$$
$$V(C_q) = \{u_1, u_2, \dots, u_q\}$$
$$E(C_q) = \{u_1u_2, u_2u_3, \dots, u_{q-1}u_q, u_qu_1\}$$
$$V(\mathcal{B}_n^3) = V(C_p) \cup V(C_q); v_i = u_i | i = 1,2,..,l$$
$$E(\mathcal{B}_n^3) = E(C_p) \cup E(C_q); v_i = u_i | i = 1,2,..,l$$

Example 9. For illustration, **Figure 13** provides an example of bicyclic graph type III, denoted as $\mathcal{B}_7^3(6,5,6)$ where $p = q = 6$. The sets of vertices and edges of bicyclic graph are

$V(C_p) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(C_p) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$ and

$V(C_q) = \{u_1, u_2, u_3, u_4, u_5u_6\}, E(C_q) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_1\}$.

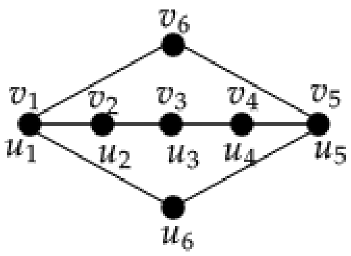


Figure 13. An example of bicyclic graph type III $\mathcal{B}_7^3(6,5,6)$

Figure 13 illustrate $\mathcal{B}_7^3(6,5,6)$, the sets of vertices and edges are

$V(\mathcal{B}_7^3(6,5,6)) = \{v_1, v_2, v_3, v_4, v_5, v_6, u_6\}$ and

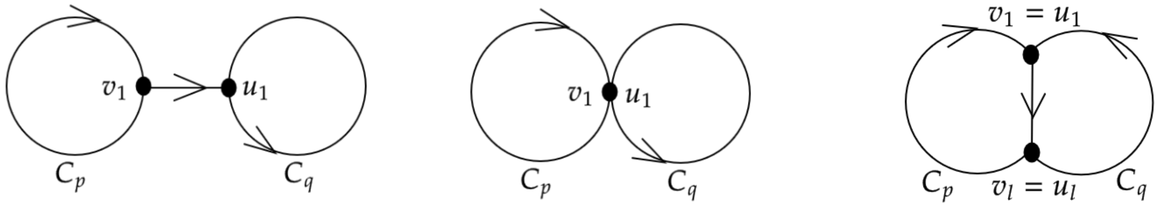
$E(\mathcal{B}_7^3(6,5,6)) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, u_5u_6, u_6u_1\}$.

Based on the definition, there are three types of bicyclic graphs: $(\mathcal{B}_n^1(p,q))$, $(\mathcal{B}_n^2(p,q))$, and $\mathcal{B}_n^3(p,l,q)$. The ∞-graph and θ-graph are associated with $(\mathcal{B}_n^1(p,q))$ and $(\mathcal{B}_n^2(p,q))$, respectively. This means that if we have a bicyclic graph of the form $(\mathcal{B}_n^1(p,q))$ and $(\mathcal{B}_n^2(p,q))$, it can be referred to as an ∞-graph. Similarly, a bicyclic graph of the form $\mathcal{B}_n^3(p,l,q)$ can be referred to as a θ-graph.

The specific structure of these graphs may vary based on the values of p, q, and l. However, the general idea is that $(\mathcal{B}_n^1(p,q))$ and $(\mathcal{B}_n^2(p,q))$ represents an ∞-shaped graph, while $\mathcal{B}_n^3(p,l,q)$ represents a θ-shaped graph. Overall, the relation suggests that different types of bicyclic graphs have distinct shapes and are labeled as ∞-graph, θ-graph, or potentially other designations based on their specific structures.

3.3 Application of Directed Bicyclic Graph

A directed bicyclic graph is a specific type of directed graph that contains two cycles. It is characterized by having two distinct cycles within its structure. These cycles can be traversed in a specific direction, with each cycle forming a closed loop. The presence of these two cycles distinguishes a directed bicyclic graph from other types of directed graphs. Illustration of directed bicyclic graph can be seen on **Figure 14**.



(a) (b) (c)

Figure 14. Ilustration of directed bicyclic graph, (a) Directed bicyclic graph type I, (b) Directed bicyclic graph type II, (c) Directed bicyclic graph type III

The construction of directed bicyclic graphs can be observed in the Trans Jogja route. Trans Jogja is a reform of the public transportation system in the Province of DIY in the form of BRT (Bus Rapit Transit) both in terms of management and the provision of adequate transportation facilities, through the concept of transportation management based on Buy the Service [23]. There are 17 Trans Jogja routes spread across in DIY. Every route is given a different color as shown in **Figure 15**.

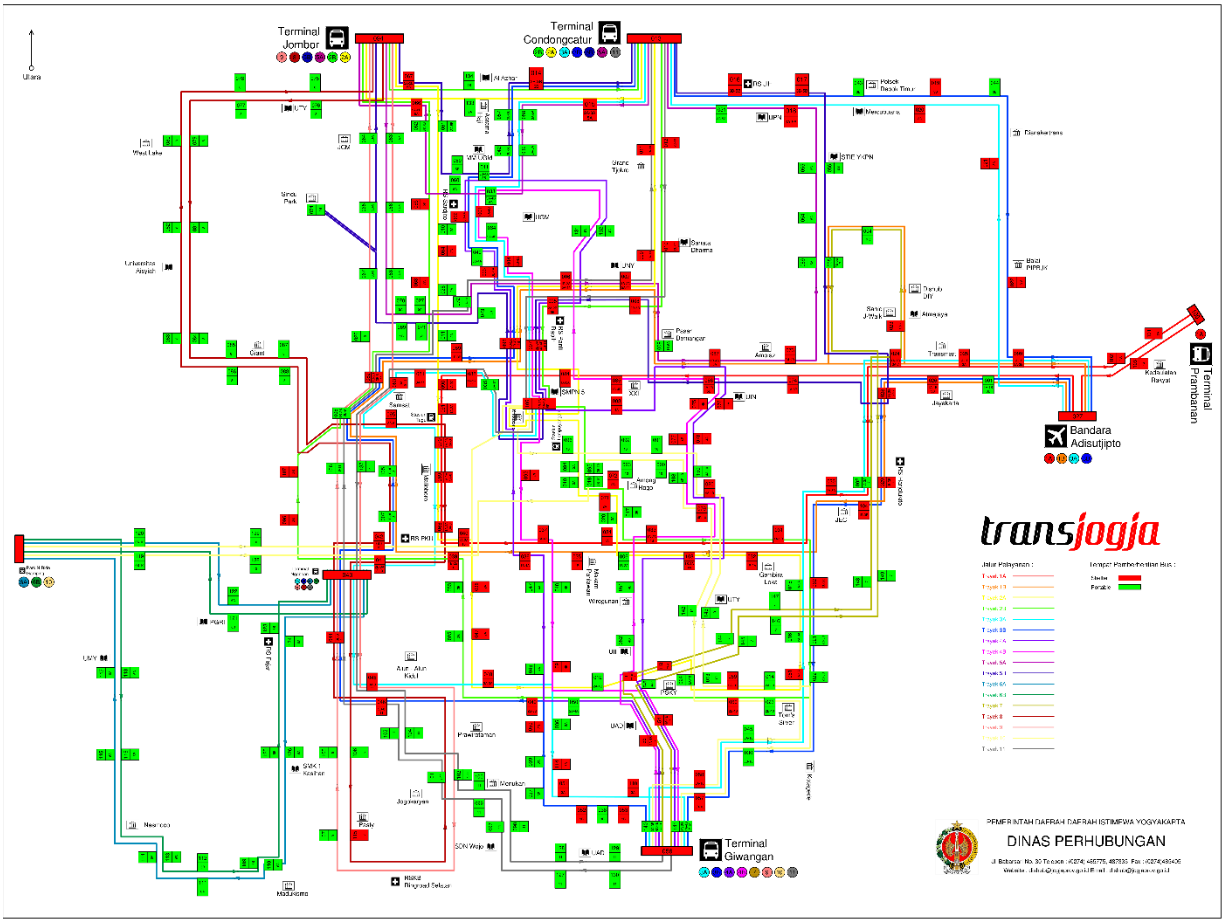


Figure 15. Trans Jogja routes

We can see that each Trans Jogja bus route forms a cycle and each route does not pass through all available stops. The bicyclic graph can be used to connect the Trans Jogja bus stops on one route to other routes. So, we can use Trans Jogja to travel from one bus stop to another in different routes. We give examples 3 types of directed bicyclic graph in Trans Jogja routes, as follows:

Example 10. The application of bicyclic graph Type I in Trans Jogja routes is illustrated as routes from UNY to UMY. Since UNY and UMY are on different routes, a single route cannot be used for travel between them. Route 11, passing through UNY, forms one directed cyclic graph, and the same applies to Route 6A passing through UMY. To travel from UNY to UMY, a route connecting the two is required, as depicted in **Figure 16.** The notations $v_i; i = 1,2, \ldots 9$ are Trans Jogja bus stop of Route 11, $u_j; j = 1,2, \ldots 7$ are Trans Jogja bus stop of Route 6A, and $x_k; k = 1,2,3$ are Trans Jogja bus stop of Route 10 that connect Route 11 and Route 6A. UNY is denoted by v_6 and UMY is denoted by u_4.

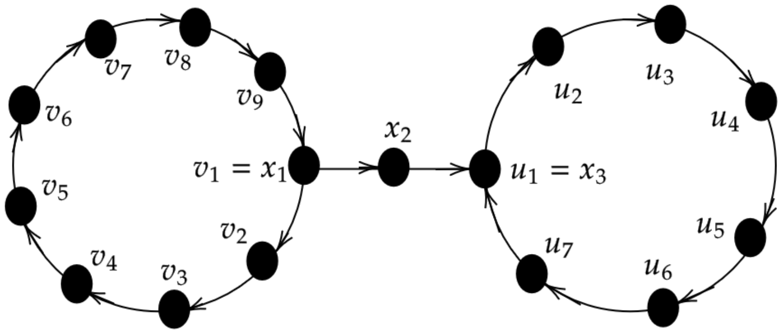


Figure 16. Example bicyclic graph Type I on Trans Jogja routes UNY to UMY

Example 11. The construction of bicyclic Type II in Trans Jogja routes can be seen in **Figure 17**. Two routes, Route 3A and Route 5A, are connected through a single Trans Jogja bus stop, Kentungan. The notations $v_i; i = 1,2, \ldots ,9$ are Trans Jogja bus stop of Route 5A and $u_j; j = 1,2, \ldots ,10$ are Trans Jogja bus stops of Route 3A. Kentungan in Route 5A is denoted by v_1 and it is denoted by u_1 in Route 3A.

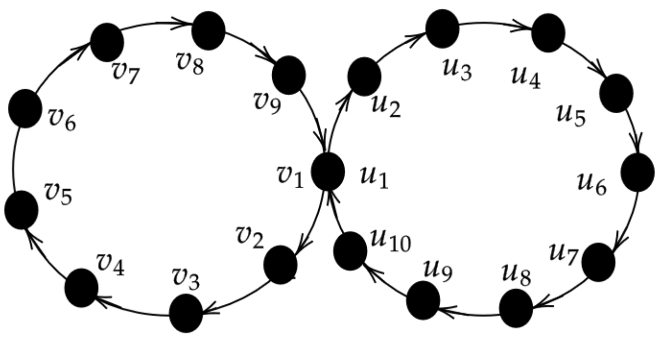


Figure 17. Example bicyclic graph Type II on Trans Jogja routes

Example 12. The construction of bicyclic graph Type III in Trans Jogja routes can be seen in **Figure 18**. Two routes, Route 2B and Route 9, are connected through one route, namely Pingit to Terminal Jombor. The notations $v_i; i = 1,2, \ldots ,7$ are Trans Jogja bus stops of Route 9 and $u_j; j = 1,2, \ldots ,10$ are Trans Jogja bus stops of Route 2B. In Route 9, Pingit and Terminal Jombor are denoted by v_1 and v_2, whereas in Route 2B, they are denoted by u_1 and u_2.

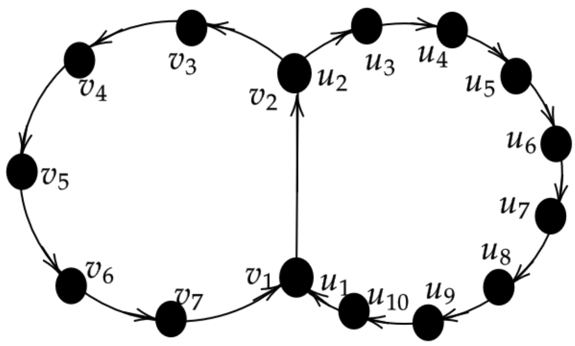


Figure 18. Example bicyclic graph Type III on Trans Jogja routes

4. CONCLUSIONS

Based on the result and discussion, the conclusion that can be drawn from this research are as follows:

1. In the beginning, there 2 types of bicyclic graph, namely: ∞-graph and θ-graph. Furthermore, these types evolved into three types: bicyclic graph Type I, Type II and Type III, denoted by $\left(\mathcal{B}_n^1(p,q)\right)$, $\left(\mathcal{B}_n^2(p,q)\right)$, and $\mathcal{B}_n^3(p,l,q)$.
2. The relation between the ∞-graph and θ-graph with bicyclic graph Type I, Type II and Type III is as follows: the ∞-graph can be devided into bicyclic graph Type I and Type II, while the θ-graph corresponds to bicyclic graph Type III.
3. Trans Jogja routes can be constructed in 3 types of bicyclic graph by adding directions to edges. Three constructions of bicyclic graph in Trans Jogja routes can be seen in **Figure 16**, **Figure 17**, and **Figure 18**.

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