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ALGEBRAIC STRUCTURES ON A SET OF DISCRETE DYNAMICAL SYSTEM AND A SET OF PROFILE

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ABSTRACT

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A discrete dynamical system is represented as a directed graph with graph nodes called states that can be seen on the dynamical map. This discrete dynamical system is symbolized by (A, g), where A is a finite set of states and the function $g: A \to A$ is a function from A to A. In the dynamical map, the discrete dynamical system has a height where the number of states in each height is called a profile. The set of discrete dynamical systems has an addition operation defined as a disjoint union on the graph and a multiplication operation defined as a tensor product on the graph. The set of discrete dynamical systems and the set of profiles are very interesting to observe from the algebraic point of view. Considering the operation of the set of discrete dynamical systems and the set of profiles, we can see their algebraic structure. By recognizing the algebraic structure, it will be easy to solve the polynomial equation in the discrete dynamical system and in the profile. In this research, we will investigate the algebraic structure of discrete dynamical systems and the set of profiles. This research shows that the set of discrete dynamical systems has an algebraic structure, which is a commutative semiring, and the set of profiles has an algebraic structure, which is a commutative semiring and $\mathbb{N}_{\geq 0}$-semimodule. Moreover, both sets have the same property, which is isomorphic to the set of non-negative integers.

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1. INTRODUCTION

A discrete dynamical system is a method that can describe the process, behavior, and complexity in a system that models a phenomenon [1]. A discrete dynamical system can be treated by numerical simulation where the solutions can be represented on a dynamic map. A discrete dynamical system is further symbolized by (A, g), where A is a finite set of states and the function $g: A \to A$ is is a function from A to A. A discrete dynamical system is the form of a category **D**, where the objects are $(A, g), (B, h), \ldots$ and the arrow is $(A, g) \to (B, h)$ given by the function $\varphi: A \to B$ [2]. The discussion of categories can be explored further outside of this research.

The discrete dynamical system (A, g) is defined as a directed graph that has a certain height that can be seen on the dynamical map. A graph is composed of a number of objects $V = v_1, v_2, v_3, \ldots$ known as nodes, which are connected to one another by links $E = e_1, e_2, e_3, \ldots$ known as edges [3]. The graph's nodes will henceforth be referred to as states. The discrete dynamical system has a specific height on the dynamic map. Limit cycles are used to describe the discrete dynamical system's basic height. A limit cycle with one state is an identity mapping that can be written as 1_A [4]. The limit cycle is at the innermost height of the dynamical map where it will be connected to other states at different heights. In this dynamic map, Brouwer's Fixed Point theorem applies, that is, the point x in A where $g(x) = x$ is called a fixed point [5]. The height of a discrete dynamical system is symbolized by $h_A(x)$ or $h(x)$. The sequence of the state of each height in the dynamical system starting from limit cycles is called profile which can be written as $\text{prof}(A, g) = (|A_0|, |A_1|, \ldots, |A_h|)$ [6]. In other words, a profile is a subset of a discrete dynamical system.

By knowing the operations possessed by the set of discrete dynamical systems and the profile set, the algebraic structure can be investigated. An algebraic structure is a non-empty set A that contains operations from A to A with one or more binary operation ([7],[8]), with operations between elements following certain rules [9] Algebraic structures have many types, including groups, rings, modules, semirings, and semimodules. Each form of algebraic structure has its own definition. By knowing the algebraic structure of a set, then we can further examine the properties possessed by the set.

Research on the algebraic structure of dynamical systems has been carried out by several researchers, although this topic is very rarely found in some literatures. A research by Guzman describes the algebraic structure of Lie algebras on non-linear dynamical systems, with Lie algebras containing Heinsenberg algebras [10]. Moreover, there are several researches on the algebraic structure of the set of discrete dynamical systems. A research by Murua and Sanz-Serna [11] describes algebraic techniques have been developed to solve several problems in the theory of discrete and continuous dynamical systems by using Hopf algebra, which is a monomorphism Lie Algebra. A research by de Castro and Kang [12] describes A groupoid with a C^* algebraic structure that is isomorphic to the C^* algebra of Boolean dynamical systems. Research by Qaralleh and Mukhamedov [13] explains the prediction of the dynamic behavior of Volterra QSOs (Quadratic Stochastic Operator) by examining the algebraic structure of the Volterra evolution algebra. QSO is used to investigate the properties of dynamical systems and modeling of dynamical systems for numerous areas. Tomiyama's research [14] explains that topological dynamical systems are homeomorphisms of C^* algebra. A research by Juan A. Aledo et al [15] explain that the dynamic behavior of Parallel Dynamical Systems (PDS) on graphs when the PDS has a Boolean algebraic structure. Dawood Khan et al's research ([16],[17]) on discrete dynamical systems which are BCI-Algebra and BCK-Algebra, investigated the properties of discrete dynamical systems on BCI-Algebra which BCI-Algebra and BCK-algebra are algebraic structures in universal algebra.

The journals described above examine the application of algebraic structures to discrete dynamical systems. It is already known that the discrete dynamical system is a category, and category theory can be proved about the algebraic structure that exists in the discrete dynamical system. Therefore, this research will focus on the theory of algebraic structures, where the proof of the complete algebraic structure will be shown on the set of discrete dynamical systems and the set of profiles. This research will show the addition and multiplication operations of the dynamical system. Then, whether the set of discrete dynamical systems is a commutative semiring will be investigated. Furthermore, the profile set will be introduced to the discrete dynamical system which will be investigated whether the profile set is a commutative semiring and $\mathbb{N}_{\geq 0}$-semimodule. Furthermore, the properties of discrete dynamical system set and profile set will also be investigated.

2. RESEARCH METHODS

The method used in this research is a literature study conducted by reviewing books, journals, and other papers on algebraic structures in dynamic systems and profile sets. The following are the steps of this research:

1. Proving that the set of dynamical systems is a semiring,
2. Proving that the set of dynamical systems is isomorphic to the set of non-negative integers,
3. Proving that the profile set is a semiring commutative,
4. Proving that the set of profiles is isomorphic to the set of non-negative integers,
5. Proving that the profile set is an $\mathbb{N}_{\geq 0}$-semimodule.

3. RESULTS AND DISCUSSION

In this section, we will discuss the algebraic structure of the set of discrete dynamical systems and the set of profiles, as well as the properties of the two sets.

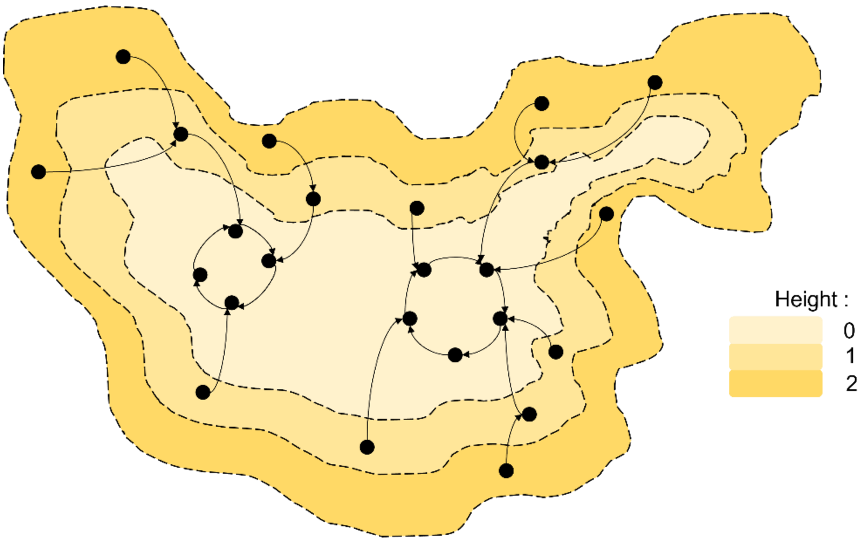


Figure 1. Maps of Discrete Dynamical Systems

Figure 1 is called the internal diagram of a set. A point inside the set represents an element. **Figure 1** is an example of a dynamic system where each height has a different state. In the discrete dynamical system, it is found that the profile is prof(7,5,9,3,3) that is, there are 7 states at height 0, 5 states at height 1, 9 states at height 2, 3 states at height 3, and 3 states at height 4. Let there exist a discrete dynamical system (A, g) with A a finite set of states and g a function $g: A \to A$ and a discrete dynamical system (B, h) with B a finite set of states and h a function $h: B \to B$. The product of the category of a dynamical system can be seen as follows:

$$(A, g) + (B, h) = (A \uplus B, g + h) \text{ where } (g + h)(x) = \begin{cases} g(x), x \in A \\ h(x), x \in B \end{cases} \quad (1)$$

$$(A, g) \times (B, h) = (A \times B, g \times h) \text{ where } (g \times h)(a, b) = \big(g(a), h(b)\big) \quad (2)$$

are defined by disjoint union of graph dan tensor product of graph **[18]**. The disjoint union and tensor product is the operation of the set of discrete dynamical systems, which can be seen in **Table 1** and **Table 2**. **Table 1** and **Table 2** explain the addition and multiplication operations on discrete dynamical systems that have states 0,1,2,3. State 0 is denoted by the empty number set. State 1 is denoted as a graph that has one point with an arrow pointing to itself. State 2 is denoted as a graph that has two points with arrows connecting the two graphs or two graphs that are not connected to each other with arrows to themselves. Similarly, state 3 is denoted as a graph that has three nodes with arrows connecting the three graphs or the three graphs that are not connected to each other by their own arrows. The number of graph shapes in each state can be varied.

Table 1. Table $(D, +)$ on state 0,1,2,3

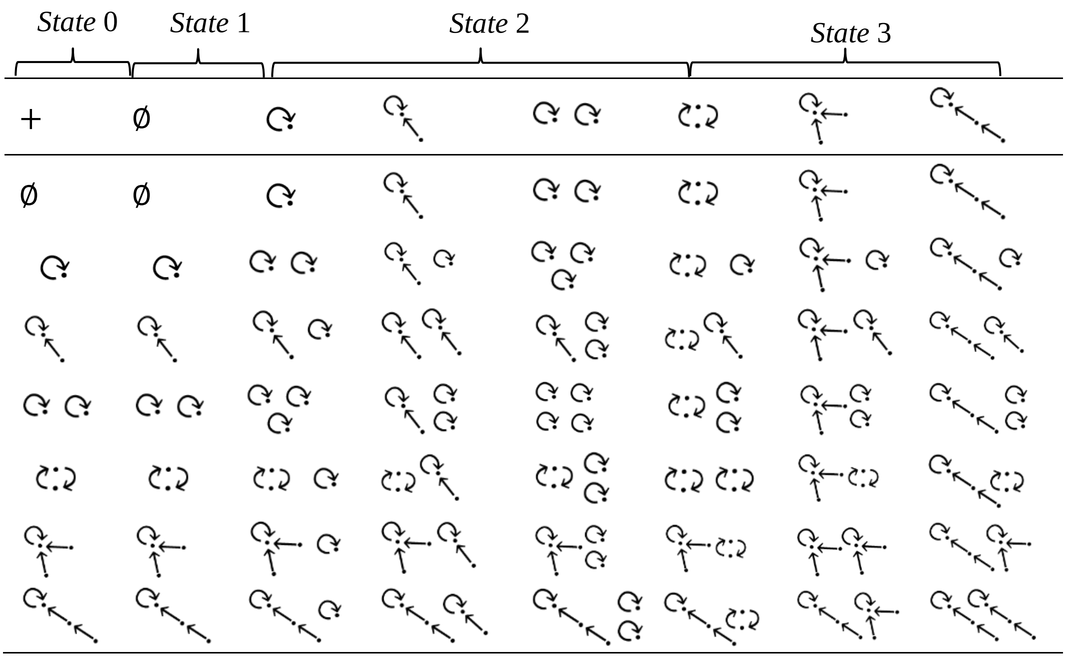
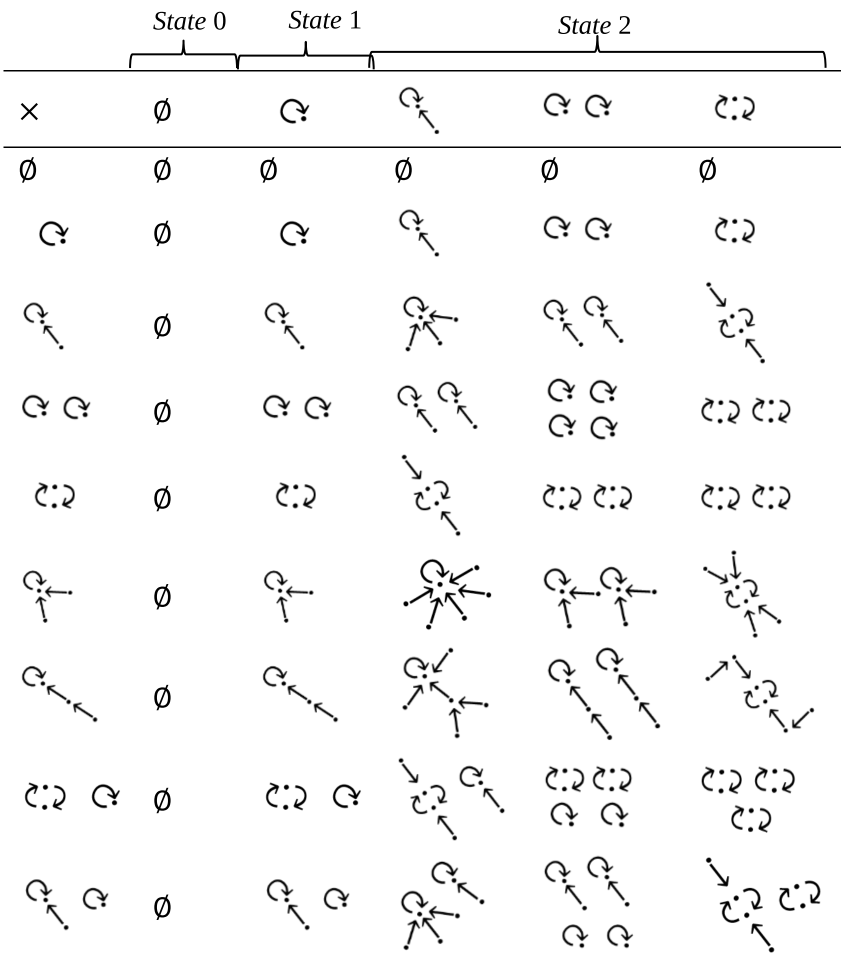
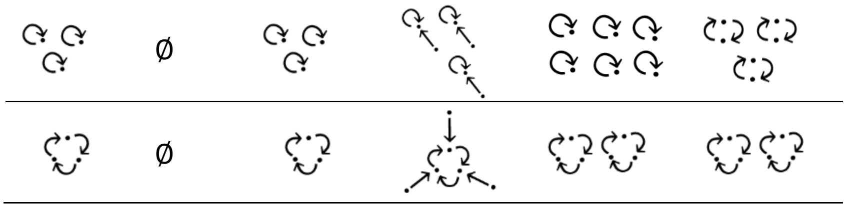


Table 2. Table of $(D, \times)$ on state 0,1,2





We have seen that the discrete dynamical system is a category, and it can be seen in the addition and multiplication table above that the set of discrete dynamical systems is a countable set. So, whether the category products of the set of discrete dynamical systems are commutative semiring. An interesting property of **D** is of the category of commutative semiring of natural numbers.

Theorem 1. (Semiring Commutative on a set of Discrete Dynamical Systems)
Let $\boldsymbol{D}$ be a set of dynamical system with $+$ and $\times$ operations, then $(\boldsymbol{D}, +, \times)$ is a commutative semiring.
Proof. First, we prove that $(\mathbf{D}, +)$ is a commutative monoid. Let $(A, g), (B, h), (C, i) \in \mathbf{D}$ with addition operation which can be seen in **Equation (1)**. Since (A, g) and $(B, h) \in \mathbf{D}$ with A and B being finite sets of states and by the definition of the summation operation in discrete dynamical systems, g and h being subsequent states of a function then $A \uplus B$ is the union of the two sets of states and $g + h$ is the disjoint sum of the two functions where for $g(x), x \in A$ and for $h(x), x \in B$ so that $(A \uplus B, g + h)$ is in $\mathbf{D}$, then $(\mathbf{D}, +)$ is closed. Next, it will be shown that $(\mathbf{D}, +)$ has commutative dan associative properties, and has a neutral element which can be seen below:

(i) $\big((A, g) + (B, h)\big) + (C, i) = (A \uplus B, g + h) + (C, i)$
Based on **Equation (1).**, then $(A, g) + (B, h) = (A \uplus B, g + h)$, then it also applied to $(A \uplus B, g + h) + (C, i)$, o that
$$\begin{aligned}\big((A, g) + (B, h)\big) + (C, i) &= (A \uplus B \uplus C, g + h + i)\\ &= (A, g) + (B \uplus C, h + i)\\ &= (A, g) + \big((B, h) + (C, i)\big)\end{aligned}$$
(ii) $$\begin{aligned}(A, g) + (B, h) &= (A \uplus B, g + h)\\ &= (B \uplus A, h + g)\\ &= (B, h) + (A, g)\end{aligned}$$
(iii) Neutral element $0_{\mathbf{D}} = \emptyset$

Because **D** has commutative dan associative properties, and has a neutral element, then $(\mathbf{D}, +)$ is commutative monoid.

Second, we will show that $(\mathbf{D}, \times)$ is a monoid. Let $(A, g), (B, h), (C, i) \in \mathbf{D}$. Since (A, g) and $(B, h) \in \mathbf{D}$ with A and B are finite sets of states and based on the definition of the multiplication operation in discrete dynamical systems which can be seen in **Equation (2)**, g and h are the next state in a function then $A \times B$ is the product of the two sets of states and $g \times h$ is the product of the two functions with $(g \times h)(a, b) = (g(a), h(b))$ so that $(A \times B, g \times h)$ is in $\mathbf{D}$, hence $(\mathbf{D}, \times)$ is closed. $(\mathbf{D}, \times)$ has associative properties and has a neutral element which can be seen below:

(i) $\big((A, g) \times (B, h)\big) \times (C, i) = (A \times B, g \times h) \times (C, i)$
Based on equation 2, then $(A, g) \times (B, h) = (A \times B, g \times h)$ then it also applies to $(A \times B, g \times h) \times (C, i)$, so that
$$\begin{aligned}&= (A \times B \times C, g \times h \times i)\\ &= (A, g) \times (B \times C, h \times i)\\ &= (A, g) \times \big((B, h) \times (C, i)\big)\end{aligned}$$
(ii) Neutral element $1_{\mathbf{D}}$

Because **D** has associative properties and has a neutral element $1_{\mathbf{D}}$, then $(\mathbf{D}, \times)$ is a monoid.

Third, we prove that D has distributive properties. Let $(A, g), (B, h), (C, i) \in \mathbf{D}$. Then do the sum and multiplication operation, we have:

(i) $$\begin{aligned}(A, g) \times \big((B, h) + (C, i)\big) &= (A, g) \times (B \uplus C, h + i)\\ &= ((A \times B) \uplus (A \times C), (g \times h) + (g \times i))\end{aligned}$$

$$= (A \times B, g \times h) + (A \times C, g \times i)$$
$$= (A, g) \times (B, h) + (A, g) \times (C, i)$$

(ii) $$\big((A, g) + (B, h)\big) \times (C, i) = (A \uplus B, g + h) \times (C, i)$$
$$= ((A \times C) \uplus (B \times C), (g \times i) + (h \times i))$$
$$= (A \times C, g \times i) + (B \times C, h \times i)$$
$$= (A, g) \times (C, i) + (B, h) \times (C, i)$$

Based on above, **D** has distributive properties.

Fourth, we prove that **D** has an absorbing element, that is $0(A, g) = 0 = (A, g)0$, $\forall$ $(A, g) \in \mathbf{D}$. Let $(A, g) \in \mathbf{D}$. Then we have $(A, g) \times 0_{\mathbf{D}} = (A, g) \times 0 = 0 = 0 \times (A, g) = 0_{\mathbf{D}} \times (A, g)$. Based on the operation above, then it is proven that $0(A, g) = 0 = (A, g), \forall\ (A, g) \in \mathbf{D}$. Based on the four axioms above, we get that $(\mathbf{D}, +, \times)$ is semiring.

Furthermore, we will show that $(\mathbf{D}, +, \times)$ is a commutative semiring. Let $(A, g), (B, h) \in \mathbf{D}$. Then we have

$$(A, g) \times (B, h) = (A \times B, g \times h)$$
$$= (B \times A, h \times g)$$
$$= (B, h) \times (A, g)$$

Based on $(A, g) \times (B, h) = (B, h) \times (A, g)\ \forall (A, g), (B, i) \in \mathbf{D}$, then it's proven that $(\mathbf{D}, +, \times)$ is a commutative semiring.

The set of discrete dynamical systems has a property, namely isomorphic to a set of non-negative integers which can be seen in **Proposition 2**.

Proposition 2. (Isomorphism of Discrete Dynamical Systems)
Let **D** *be a set of dynamical system. Semiring* **D** *has a subsemiring* **N** *isomorphic to a set of non-negative integer*.

Proof. Let $\varphi: \mathbb{N}_{\geq 0} \to \mathbf{D}$ a dynamical system containing n fixed points (i.e. identity function over the set of n points) and suppose that, $\mathbf{N} = \varphi(\mathbb{N}_{\geq 0})$ with $\mathbf{0} = \varphi(0)$ and $\mathbf{1} = \varphi(1)$. First it will be shown that φ is a homomorphism semiring, i.e. it will be shown that $\varphi(n + m) \in \mathbf{N}$ and $\varphi(n \times m) \in \mathbf{N}$, $\forall \varphi(n), \varphi(m) \in \mathbf{N}$. Suppose two arbitrary elements are operated on with the elements being $\varphi(n), \varphi(m) \in \mathbf{N}$ with $\varphi(n) = \mathbf{n}$ and $\varphi(m) = \mathbf{m}$. Then the elements are operated as follows:

(i) $\varphi(m) + \varphi(n) = \mathbf{m} + \mathbf{n} = \varphi(m + n)$ with $\varphi(m + n) \in \mathbf{N}$

(ii) $\varphi(m) \times \varphi(n) = \mathbf{m} \times \mathbf{n} = \varphi(m \times n)$ with $\varphi(m \times n) \in \mathbf{N}$

Then for all $\varphi(n), \varphi(m) \in \mathbf{N}$**,** we have $\varphi(m + n) = \varphi(m) + \varphi(n)$ dan $\varphi(m \times n) = \varphi(m) \times \varphi(n)$ which $\varphi(m + n), \varphi(m \times n) \in \mathbf{N}$. Based on the above proof, φ is a semiring homomorphism.

Second, we prove that φ semiring monomorphism. Based on the above operation, we have $\varphi(n) = \mathbf{n}$ and $\varphi(m) = \mathbf{m}$. If $\varphi(m) = \varphi(n)$ then $\mathbf{m} = \mathbf{n}$ causes injective so that it is semiring monomorphism. The two points above prove that a semiring **D** containing an isomorphic subsemiring **N** with $\mathbb{N}_{\geq 0}$.

After knowing the algebraic structure and properties of the set of dynamical systems, we will also investigate the algebraic structure of the profile set. Keep in mind that the profile set is part of the set of discrete dynamical systems so that the algebraic structure and properties of the profile set will tend to be the same as the set of discrete dynamical systems. Before investigating the structure, it will be shown how the operations that apply to the set of profiles. Given two profiles $\boldsymbol{p} = (p_i)_{i \in \mathbb{N}}$ and $\boldsymbol{q} = (q_i)_{i \in \mathbb{N}}$. The summation of the two profiles is as follows

$$\boldsymbol{p} + \boldsymbol{q} = (p_i + q_i)_{i \in \mathbb{N}}. \tag{3}$$

The multiplication of two profiles is as follows:

$$\mathbf{p} \times \mathbf{q} = \left(p_i \times \sum_{j=0}^{i} q_i + q_i \times \sum_{j=0}^{i} p_i - p_i \times q_i \right)_{i \in \mathbb{N}} \tag{4}$$

The illustration of addition and multiplication in Equation (3) and Equation (4) can be seen in Table 3 where the table contains the set of profiles **P** for the sum operation and Table 4 contains the set of profiles **P** for the multiplication operation.

Table 3. Table $(\mathbf{P},+)$ on state 0,1,2,3

$+$	(0)	(1)	(2)	(1,1)	(3)	(1,2)	(2,1)	(1,1,1)	$\cdots$
(0)	(0)	(1)	(2)	(1,1)	(3)	(1,2)	(2,1)	(1,1,1)	$\cdots$
(1)	(1)	(2)	(3)	(2,1)	(4)	(2,2)	(3,1)	(2,1,1)	$\cdots$
(2)	(2)	(3)	(4)	(3,1)	(5)	(3,2)	(4,1)	(3,1,1)	$\cdots$
(1,1)	(1,1)	(2,1)	(3,1)	(2,2)	(4,1)	(2,3)	(3,2)	(2,2,1)	$\cdots$
(3)	(3)	(4)	(5)	(4,1)	(6)	(4,2)	(5,2)	(4,1,1)	$\cdots$
(1,2)	(1,2)	(2,2)	(3,2)	(2,3)	(4,2)	(2,4)	(3,3)	(2,3,1)	$\cdots$
(2,1)	(2,1)	(3,1)	(4,1)	(3,2)	(5,1)	(3,3)	(4,2)	(3,2,1)	$\cdots$
(1,1,1)	(1,1,1)	(2,1,1)	(3,1,1)	(2,2,1)	(4,1,1)	(2,3,1)	(3,2,1)	(2,2,2)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 4. Table $(\mathbf{P},\times)$ on state 0,1,2,3

$\times$	(0)	(1)	(2)	(1,1)	(3)	(1,2)	(2,1)	(1,1,1)	$\cdots$
(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	$\cdots$
(1)	(0)	(1)	(2)	(1,1)	(3)	(1,2)	(2,1)	(1,1,1)	$\cdots$
(2)	(0)	(2)	(4)	(2,2)	(6)	(2,4)	(4,2)	(2,2,2)	$\cdots$
(1,1)	(0)	(1,1)	(2,2)	(1,3)	(3,3)	(1,5)	(2,4)	(1,3,2)	$\cdots$
(3)	(0)	(3)	(6)	(3,3)	(9)	(3,6)	(6,3)	(3,3,3)	$\cdots$
(1,2)	(0)	(1,2)	(2,4)	(1,5)	(3,6)	(1,8)	(2,7)	(1,5,3)	$\cdots$
(2,1)	(0)	(2,1)	(4,2)	(2,4)	(6,3)	(2,7)	(4,5)	(2,4,3)	$\cdots$
(1,1,1)	(0)	(1,1,1)	(2,2,2)	(1,3,2)	(3,3,3)	(1,5,3)	(2,4,3)	(1,3,5)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

After knowing about the addition and multiplication operations, we will then investigate the algebraic structure of the set of profiles.

Theorem 3. (Commutative Semiring on a Set of Profile)
If $\mathbf{P}$ be a set of profile with $+$ and $\times$ operations, then $(\mathbf{P},+,\times)$ is a commutative semiring.
Proof. First, we prove that $(\mathbf{P},+)$ is a commutative monoid. Let $\text{prof}(A,g),\text{prof}(B,h)\in\mathbf{P}$, so that by Proposition 2 it is obtained that semiring **D** is isomorphic to the set of non-negative numbers so that the sum of two profiles whose members are the set of non-negative integers will produce a member of the set of non-negative integers, then $(\mathbf{P},+)$ is closed. Based on Proposition 2, for the + operation, P inherits the associative and commutative properties of $\mathbb{N}_{\geq 0}$. **P** has neutral element $0_{\mathbf{P}}=(0)$ and has neutral element $0_{\mathbf{D}}=\emptyset$. Because **P** has commutative dan associative properties, and has a neutral element, then $(\mathbf{P},+)$ is commutative monoid.

Second, we show that $(\mathbf{P},\times)$ is a monoid. Let $\mathbf{p},\mathbf{q},\mathbf{r}\in\mathbf{P}$ and let $(A,g),(B,h),(C,i)\in\mathbf{D}$ with $\text{prof}(A,g)=\mathbf{p},\text{prof}(B,h)=\mathbf{q}$, and $\text{prof}(C,i)=\mathbf{r}$. $(\mathbf{P},\times)$ has associative properties and has a neutral element which can be seen below:

(i) $$\begin{aligned}(\mathbf{p}\times\mathbf{q})\times\mathbf{r} &= \text{prof}((A,g)\times(B,h))\times\text{prof}(C,i)\\ &= \big(\text{prof}(A,g)\times\text{prof}(B,h)\big)\times\text{prof}(C,i)\\ &= \text{prof}(A,g)\times\text{prof}(B,h)\times\text{prof}(C,i)\\ &= \text{prof}(A,g)\times\big(\text{prof}(B,h)\times\text{prof}(C,i)\big)\\ &= \text{prof}(A,g)\times\text{prof}\big((B,g)\times(C,i)\big)\\ &= \mathbf{p}\times(\mathbf{q}\times\mathbf{r})\end{aligned}$$

(ii) Neutral element $1_{\mathbf{P}}=\text{prof }1_{\mathbf{D}}=(1)$

Because **P** has associative properties and has a neutral element $1_{\mathbf{P}}$, then $(\mathbf{P},\times)$ is a monoid.

Third, we prove that **P** has distributive properties. Let $\mathbf{p},\mathbf{q},\mathbf{r}\in\mathbf{P}$.Then do the sum and multiplication operation, we have:

(i) $\mathbf{p} \times (\mathbf{q} + \mathbf{r}) = \text{prof}(A, g) \times \left(\text{prof}\big((B, h) + (C, i)\big)\right)$
$= \text{prof}(A, g) \times \big(\text{prof}(B, h) + \text{prof}(C, i)\big)$
$= \text{prof}(A, g) \times \text{prof}(B, h) + \text{prof}(A, g) \times \text{prof}(C, i)$
$= \text{prof}\big((A, g) \times (B, h)\big) + \text{prof}\big((A, g) \times (C, i)\big)$
$= (\mathbf{p} \times \mathbf{q}) + (\mathbf{p} \times \mathbf{r})$

(ii) $(\mathbf{p} + \mathbf{q}) \times \mathbf{r} = \left(\text{prof}\big((A, g) + (B, h)\big)\right) \times \text{prof}(C, i)$
$= \big(\text{prof}(A, g) + \text{prof}(B, h)\big) \times \text{prof}(C, i)$
$= \text{prof}(A, g) \times \text{prof}(C, i) + \text{prof}(B, h) \times \text{prof}(C, i)$
$= \text{prof}\big((A, g) \times (C, i)\big) + \text{prof}\big((B, h) \times (C, i)\big)$
$= (\mathbf{p} \times \mathbf{r}) + (\mathbf{q} \times \mathbf{r})$

Based on the operation above, **P** has distributive properties.

Fourth, we prove that $0\mathbf{p} = 0 = \mathbf{p}0, \forall\, \mathbf{p} \in \mathbf{P}$. Let $\mathbf{p} \in \mathbf{P}$. Then we have $\mathbf{p} \times 0_{\mathbf{p}} = \mathbf{p} \times (0) = (0) = (0) \times \mathbf{p}$. Based on the operation above, then it's proven that $0\mathbf{p} = 0 = \mathbf{p}0, \forall\, \mathbf{p} \in \mathbf{P}$. Based on the four axioms above, then we get that $(\mathbf{P}, +, \times)$ is semiring.

Furthermore, proof that $(\mathbf{P}, +, \times)$ is a commutative semiring. Let $\mathbf{p}, \mathbf{q} \in \mathbf{P}$. Then we have

$\mathbf{p} \times \mathbf{q} = \text{prof}(A, g) \times \text{prof}(B, h)$
$= \text{prof}\big((A, g) \times (B, h)\big)$
$= \text{prof}\big((B, h) \times (A, g)\big)$
$= \text{prof}(B, h) \times \text{prof}(A, g)$
$= \mathbf{q} \times \mathbf{p}$

Based on $\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{p}\ \forall \mathbf{q}, \mathbf{p} \in \mathbf{P}$, then it's proven that $(\mathbf{P}, +, \times)$ is a commutative semiring.

Similar to the set of discrete dynamical systems, the profile set has a property, which is isomorphic to a set of non-negative integers which can be seen in Lemma 4.

Lemma 4. (Isomorphism of Profile)
*Let **P** be a set of profile. $(\mathbf{P}, +, \times)$ has subsemiring that is isomorphic to a set of non-negative integer.*
Proof. Let $\varphi: \mathbb{N}_{\geq 0} \to \mathbf{P}$ and $\varphi(n) = (n)$ with $\varphi(0) = (0)$ and $\varphi(1) = (1)$. First, it is proved that φ is a homomorphism semiring, i.e. it will be shown that $\varphi(n + m) \in \mathbf{D}$ and $\varphi(n \times m) \in \mathbf{D}\ \forall \varphi(n), \varphi(m) \in \mathbf{D}$. Let two arbitrary elements are operated on with the elements being $\varphi(n), \varphi(m) \in \mathbf{D}$ with $\varphi(n) = (n)$ and $\varphi(m) = (m)$. Then the elements are operated as follows:

(i) $\varphi(m) + \varphi(n) = (m) + (n) = (m + n) = \varphi(m + n)$ with $\varphi(m + n) \in \mathbf{D}$
(ii) $\varphi(m) \times \varphi(n) = (m) \times (n) = (m \times n) = \varphi(m \times n)$ with $\varphi(m \times n) \in \mathbf{D}$

Then for all $\varphi(n), \varphi(m) \in \mathbf{D}$, we have $\varphi(m + n) = \varphi(m) + \varphi(n)$ and $\varphi(m \times n) = \varphi(m) \times \varphi(n)$ which $\varphi(m + n), \varphi(m \times n) \in \mathbf{D}$.

Based on the above proof, φ is a semiring homomorphism.

Second, we prove that φ semiring monomorphism. Based on the above operation, we have $\varphi(n) = (n)$ and $\varphi(m) = (m)$. If $\varphi(m) = \varphi(n)$ then $(m) = (n)$ causes injective so that it is semiring monomorphism. The two points above prove that image of $\varphi(\mathbb{N}_{\geq 0})$ is an isomorphic subsemiring of **P** with $\mathbb{N}_{\geq 0}$.

Besides having the algebraic structure of a commutative semiring, it is also investigated that the profile set is an $\mathbb{N}_{\geq 0}$-semimodule which can be seen in **Theorem 5.**

Theorem 5. ($\mathbb{N}_{\geq 0}$-semimodule of Profile)
*If **P** is a set of profile with sum operation and $\mathbb{N}_{\geq 0}$ is a non-negative integer, then $(\mathbf{P}, +)$ as an $\mathbb{N}_{\geq 0}$-semimodule.*
Proof. Based on Theorem 2, $(\mathbf{P}, +)$ is a commutative monoid. Therefore, it will be proved that $(\mathbf{P}, +)$ is a $\mathbb{N}_{\geq 0}$-semimodule with scalar operation $\bullet: \ \mathbf{P} \times \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ as follows:

$$\text{prof}(A, f) \bullet k = \mathbf{p}k\ \forall \mathbf{p} \in \mathbf{P} \text{ and } k \in \mathbb{N}_{\geq 0}$$

Let $\text{prof}(A, g), \text{prof}(B, h) \in \mathbf{P}$ with $\text{prof}(A, g) = \mathbf{p}$ and $\text{prof}(B, h) = \mathbf{q}$, and $k, l \in \mathbb{N}_{\geq 0}$

1) First, we show that $\text{prof}(A,g) \bullet (n+m) = \text{prof}(A,g) \bullet n + \text{prof}(A,g) \bullet m$

$$\begin{aligned}\text{prof}(A,g) \bullet (k+l) &= \mathbf{p} \bullet (k+l)\\ &= \mathbf{p} \bullet k + \mathbf{p} \bullet l\\ &= \text{prof}(A,g) \bullet k + \text{prof}(A,g) \bullet l\end{aligned}$$

Based on the above proof, it is proven that $\text{prof}(A,g) \bullet (k+l) = \text{prof}(A,g) \bullet k + \text{prof}(A,g) \bullet l$.

2) Second, we prove that $\left(\text{prof}(A,g) + \text{prof}(B,h)\right) \bullet k = \text{prof}(A,g) \bullet k + \text{prof}(B,h) \bullet k$

$$\begin{aligned}\left(\text{prof}(A,g) + \text{prof}(B,h)\right) \bullet n &= (\mathbf{p} + \mathbf{q}) \bullet k\\ &= \mathbf{p} \bullet k + \mathbf{q} \bullet k\\ &= \text{prof}(A,g) \bullet k + \text{prof}(B,h) \bullet k\end{aligned}$$

Based on the above proof, it is proven that $\left(\text{prof}(A,g) + \text{prof}(B,h)\right) \bullet k = \text{prof}(A,g) \bullet k + \text{prof}(B,h) \bullet k$.

3) Third, we show that $\text{prof}(A,g) \bullet (\text{prof}(B,h) \bullet k) = (\text{prof}(A,g) \bullet \text{prof}(B,h)) \bullet k$

$$\begin{aligned}\text{prof}(A,g) \bullet (\text{prof}(B,h) \bullet k) &= \mathbf{p} \bullet (\mathbf{q} \bullet k)\\ &= (\mathbf{p} \bullet \mathbf{q} \bullet k)\\ &= ((\mathbf{p} \bullet \mathbf{q}) \bullet k)\\ &= (\text{prof}(A,g) \bullet \text{prof}(B,h)) \bullet k\end{aligned}$$

Based on the above proof, it is proven that $\text{prof}(A,g) \bullet (\text{prof}(B,h) \bullet k) = (\text{prof}(A,g) \bullet \text{prof}(B,h)) \bullet k$.

4) Fourth, we prove that $\mathbf{1_P}k = k$

$$\begin{aligned}\mathbf{1_P}k &= (1)k\\ &= k\end{aligned}$$

Based on the above proof, it is proven that $\mathbf{1_P}k = k$.

5) Fifth, we prove that $\mathbf{0_P}k = \mathbf{0_P}$

$$\begin{aligned}\mathbf{0_P}k &= (0)k\\ &= (0)\\ &= \mathbf{0_P}\end{aligned}$$

Based on the above proof, it is proven that $\mathbf{0_P}k = \mathbf{0_p}$.

The five points above prove that $(\mathbf{P},+)$ is a $\mathbb{N}_{\geq 0}$-semimodul.

This research has shown that the set of discrete dynamical systems and the set of profiles have algebraic structures, namely commutative semiring and $\mathbb{N}_{\geq 0}$-semimodule. Moreover, further research can be done on the benefits and usage of the algebraic structure on the profile set. It can also be reviewed if it could be connected to the problem-solving on polynomial equations of discrete dynamical systems.

4. CONCLUSIONS

We show that a set of dynamical systems has an algebraic structure, namely commutative semiring. The dynamical system also has the property that the subsemiring of the dynamical system **D** is isomorphic to a non-negative integers. Moreover, we also show that a set of profiles has two structures, namely a commutative semiring and a semimodule $\mathbb{N}_{\geq 0}$. Profiles also have the property that the subsemiring of profile **P** is isomorphic to a set of non-negative integers.

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REFERENCES

[1] Hartrisari, "Sistem dinamik: konsep sistem dan permodelan untuk industri dan lingkungan," *SEAMEO Biotrop*, 2007.

[2] A. Dennunzio, V. Dorigatti, E. Formenti, L. Manzoni, and A. E. Porreca, *Polynomial Equations over Finite, Discrete-Time Dynamical Systems*, vol. 11115 LNCS. Springer International Publishing, 2018. doi: 10.1007/978-3-319-99813-8_27.

[3] L. AlSuwaidan and M. Ykhlef, "Toward Information Diffusion Model for Viral Marketing in Business," *2016*.

[4] F. W. Lawvere and S. H. Schanuel, "Conceptual mathematics: a first introduction to categories," *Choice Rev. Online*, vol. 36, no. 02, pp. 36-1018-36–1018, 2009, doi: 10.5860/choice.36-1018.

[5] J. Á. Cid and J. Mawhin, "The Brouwer fixed point theorem and periodic solutions of differential equations," *J. Fixed Point Theory Appl.*, vol. 25, no. 1, pp. 1–12, 2023, doi: 10.1007/s11784-022-01023-x.

[6] C. Gaze-maillot and A. E. Porreca, "Profiles of dynamical systems and their algebra," *arXiv2008.00843 [cs, math]*, 2020, [Online]. Available: http://arxiv.org/abs/2008.00843

[7] M. L. Kilpack, "Algebraic Structure Representations for Lattices," *arXiv Prepr. arXiv*, 2018.

[8] A. Triyani and S. A. Larasati, "Algebraic Structures of Interval Sets," 2020.

[9] J. Liesen, *Algebraic Structures*. 2015.

[10] J. R. Guzmán, "On a Lie algebraic Structure Associated with a Non-Linear Dynamical System," *arXiv Prepr.*, 2019.

[11] E. Patterson, "The algebra and machine representation of statistical models," no. June, 2020, [Online]. Available: http://arxiv.org/abs/2006.08945

[12] G. G. de Castro and E. J. Kang, "Boundary path groupoids of generalized Boolean dynamical systems and their C∗-algebras," *J. Math. Anal. Appl.*, vol. 518, no. 1, 2023, doi: 10.1016/j.jmaa.2022.126662.

[13] I. Qaralleh and F. Mukhamedov, "Volterra evolution algebras and their graphs," *Linear Multilinear Algebr.*, vol. 69, no. 12, pp. 2228–2244, 2021, doi: 10.1080/03081087.2019.1664387.

[14] J. Tomiyama, "Hulls and kernels from topological dynamical systems and their applications to homeomorphism C*-algebras," *J. Math. Soc. Japan*, vol. 56, no. 2, pp. 349–364, 2004, doi: 10.2969/jmsj/1191418634.

[15] J. A. Aledo, S. Martinez, and J. C. Valverde, "Parallel dynamical systems over graphs and related topics: A survey," *J. Appl. Math.*, vol. 2015, pp. 26–29, 2015, doi: 10.1155/2015/594294.

[16] D. Khan, A. Rehman, N. Sheikh, S. Iqbal, and I. Ahmed, "Properties of Discrete Dynamical System in BCI-Algebra," *Int. J. Manag. Fuzzy Syst.*, vol. 6, no. 3, p. 53, 2020, doi: 10.11648/j.ijmfs.20200603.12.

[17] D. Khan and A. Rehman, "A New View of Homomorphic Properties of BCK-Algebra in Terms of Some Notions of Discrete Dynamical System," *J. New Theory*, vol. 52, no. 3, pp. 95–100, 2021.

[18] K. H. Rosen, *Handbook of Discrete and Combinatorial Mathematics*. CRC press, 1999.