

SAWI TRANSFORMATION FOR SOLVING A SYSTEM OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

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There are many problems in nature whose solutions are obtained through mathematical concepts. One of the most common mathematical concepts is a mathematical concept that is classified under initial value problems, such as a system of linear ordinary differential equations equipped with initial values. One tool that can solve the initial value problem is the Sawi transformation. This article describes the study of the initial value problem as a system of linear ordinary differential equations and its solution using the Sawi transformation. In addition, as part of applying the resulting theory, 2 (two) case examples are given (a first-order chemical reaction system with three certain chemicals and a mass-spring system with forced motion) to be solved using the Sawi transformation. So that problem solving can be interpreted and easily understood, in the 2 (two) case studies discussed and focused on the concentration of the chemical reactants, simulations were carried out for several different initial values and reaction rate constants. Compared to other methods (Laplace transform), the results obtained from using the Sawi transformation for the cases discussed show that the analytical solutions for the selected initial values have similar solutions.



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1. INTRODUCTION

The integral transform method is a convenient mathematical tool. It can quickly deal with the first choice of researchers for solving advanced problems and solutions to critical issues in various sciences, for example, astronomy, economics, telecommunications, statistics, thermal sciences, marine sciences, biology, chemistry, physics, mathematics, medicine, aerodynamics, civil engineering, control theory, cardiology, and mechanics. Some real problems in the world can be expressed in multiple forms of mathematical modelling. The modellings are in the form of equations or a system of equations such as the ordinary differential equation. An ordinary differential equation is a differential equation that contains the ordinary derivative of one or more dependent variables concerning one independent variable (see [1]). In contrast, a system of differential equations is an equation consisting of two or more related equations. Systems of differential equations are classified into two systems: linear and nonlinear differential equations [2]. To solve a systems of differential equations can be used a matrix Laplace transformation method. Especially, we can use it to solve initial value problems of second-order homogenous linear systems of differential equations with constant coefficients [3].

The Sawi transformation method is a form of integral transformation obtained from the Fourier integral so that the Sawi transformation and its basic properties are obtained. This method was introduced by Mahgoub in 2019 and effectively solves problems in ordinary and partial differential equations [4]. Systems in ordinary differential equations are widely applied to various problems, such as deflection problems in curve beams, problems in three-layered beams, aircraft control problems in cosmic space, problems in chemical reaction circuits, and others [5]. There are quite several studies on the Sawi transformation, but there are still few studies on the application of the Sawi transformation in various problems in various fields. The study of the Sawi transformation in various problems was carried out by Aggarwal (2019), successfully applying the transform to the problem of population growth and decline [6]. In 2021, Sahoo applied the Sawi transformation to solve problems in Newton's law of cooling [7]. Higazy and Aggarwal in 2021 succeeded in applying the Sawi transformation to find a concentration solution from a first-order chemical reaction by considering four numerical problems [8]. Aggarwal and Gupta in 2019 established duality relations between some useful integral transformations namely Laplace transformation, Kamal transformation, Elzaki transformation, Aboodh transformation, Sumudu transformation, Mahgoub (Laplace-Carson) transformation and Mohand transformation with Sawi transformation [9]. Sawi decomposition method for the primitive of linear Faltung-type second kind linear Faltung-type Volterra integral equation is given by [10].

The development of research in the last decade on Sawi transformation has produced many new things. For example, the Sawi decomposition method combines the transformation and Sawi decomposition methods. This method can be used to determine the primitive Volterra integral equation (V.I.E.) with applications [11]–[15].

This article contains the following: in part 2, a review is given containing the definitions and characteristics of the Sawi transformation, then section 3 contains two issues that have been defined, namely the chemical reaction system of three certain chemical substances to determine their concentration and the forced motion mass-spring system. to determine the spring displacement of the two masses. Then it is described the completion of the analytic solutions of the two problems step by step using the Sawi transformation so that the analytic solutions are presented in the form of numerical tables and graphs, and section 4 contains the conclusions of the research results.

2. RESEARCH METHODS

This article was obtained based on the research initiated by observing some literature regarding Sawi's transformation theory and its research developments. A few definitions and theorems are given in this section to support the research. Besides that, the literature study provides information that one of the tools to solve the initial value problem has a similar solution procedure to the Laplace transform, known as the Sawi transformation. To solve the initial value problem in the form of a system of linear ordinary differential equations, we show that the Sawi transformation can be used if several additional modifications and identity forms appropriated to the problem are given. We provide it in this section with examples of applying the resulting theory. Two examples (a first-order chemical reaction system with three specific chemicals and a mass-spring system with forced motion) are also involved. For numerical work, we focused

on the concentration of chemical reactants to solve a problem that can be interpreted and easily understood for several different initial values and reaction rate constants. Compared to other methods (Laplace transformation), the results obtained from using the Sawi transformation for the cases discussed show that the analytical solutions for the selected initial values have similar solutions.

2.1 Definition of Sawi Transformation

Mahgoub (2019) define that if given a set A with its members is an exponential function with power so that it can be written [4],

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0. |f(t)| < Me^{\frac{|t|}{k_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\} \quad (1)$$

With M is a finite number and k_1, k_2 is a finite or infinite number. Then the Sawi transformation is $f(t)$ denoted by the operator $S(\cdot)$ is

$$S\{f(t)\} = T(\mu) = \frac{1}{\mu^2} \int_0^{\infty} f(t) e^{-\frac{t}{\mu}} dt, t \geq 0, k_1 \leq \mu \leq k_2 \quad (2)$$

2.2 Properties of the Sawi Transformation

The Sawi transformation has properties with the following definitions (see [8] and [16] for detail).

1. The linearity property of the Sawi transformation is defined that if $S\{f_1(t)\} = T_1(\mu)$ and $S\{f_2(t)\} = T_2(\mu)$, then $S[mf_1(t) + nf_2(t)] = [mT_1(\mu) + nT_2(\mu)]$. Where m, n is an arbitrary constants.
2. The scaling properties of the Sawi transformation are defined as follows
 - a. If $T(\mu)$ is the Sawi transformation of $f(t)$ then $kT(k\mu)$ is the Sawi transformation of the function $f(kt)$.
 - b. The translation properties of the Sawi transformation are defined that if the Sawi transformation

$$T(\mu) \text{ is } \frac{1}{(1-k\mu)^2} T\left(\frac{\mu}{1-k\mu}\right) \text{ of the Sawi transform of the function } e^{kt} f(t)$$

2.3 The Derivative Function of the Sawi Transformation

The derivative function of the Sawi transformation is defined as if $T(\mu)$ Sawi transform of $f(t)$, then

1. $\frac{1}{\mu} T(\mu) - \frac{1}{\mu^2} f(0)$ is the Sawi transformation of the derivative function $f'(t)$.
2. $\frac{1}{\mu^2} T(\mu) - \frac{1}{\mu^3} f(0) - \frac{1}{\mu^2} f'(0)$ is the Sawi transformation of the derivative function $f''(t)$.
3. $\frac{1}{\mu^\rho} T(\mu) - \sum_{k=0}^{\rho-1} \left(\frac{1}{\mu}\right)^{\rho-(k-1)} f^{(k)}(0)$ is the Sawi transformation of the derivative function $f^{(\rho)}(t)$.

2.4 The Duality of the Laplace-Sawi Transformation

In 2019, the duality of the Laplace-Sawi transformation is defined by Aggarwal and Gupta. If the Laplace transformation of $Z(\alpha)$ is $F(\mu)$ and the Sawi transformation of $Z(\alpha)$ is $T(\mu)$, then [9]

$$F(\mu) = \frac{1}{\mu^2} T\left(\frac{1}{\mu}\right) \quad (3)$$

$$T(\mu) = \frac{1}{\mu^2} F\left(\frac{1}{\mu}\right) \quad (4)$$

2.5 The Inverse of the Sawi Transformation

The inverse Sawi transformation is the inverse process of the Sawi transformation which is defined as follows: Suppose that $T(\mu)$ a function from the Sawi transform $f(t)$ has the property $S\{f(t)\} = T(\mu)$, then $f(t)$ is called the inverse Sawi transformation of $T(\mu)$. This relationship can be seen in **Table 1**.

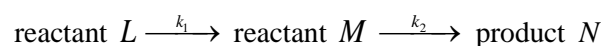
Table 1. Sawi Transformation From Several Functions $f(t)$

$f(t)$	$S\{f(t)\}$	$f(t)$	$S\{f(t)\}$
1	$\frac{1}{\mu}$	$e^{\beta t}$	$\frac{1}{\mu(1-\beta\mu)}$
t	1	$\sin \beta t$	$\frac{\beta}{(1+\beta^2\mu^2)}$
t^2	2μ	$\cos \beta t$	$\frac{1}{\beta(1+\beta^2\mu^2)}$
$t^\rho, \rho \in N$	$\rho!(\mu)^{\rho-1}$	$\sinh \beta t$	$\frac{\beta}{(1-\beta^2\mu^2)}$
$t^\rho, \rho > -1$	$\Gamma(\rho+1)(\mu)^{\rho-1}$	$\cosh \beta t$	$\frac{1}{\beta(1-\beta^2\mu^2)}$

3. RESULTS AND DISCUSSION

3.1 Numerical Example: System of Chemical Reactions of Three Specific Chemical Substances

Consider three specific chemical substances in a row: chemicals L, M , and N . The concentration of chemical L is C_1 , the concentration of a chemical M is C_2 , and the concentration of a chemical N is C_3 . The chemical reaction assisted by mixing a chemical substance that reacts with each reactant L, M , and product N and reacts at a rate k_1, k_2 which assumes that $k_1, k_2 > 0$. The chemical reaction illustrated as in the chemical reaction below [5].



So that the chemical reaction system modeled in the form of the following system of ordinary differential equations.

$$\begin{aligned} \frac{dC_1}{dt} &= -k_1 C_1 \\ \frac{dC_2}{dt} &= k_1 C_1 - k_2 C_2 \\ \frac{dC_3}{dt} &= k_2 C_2 \end{aligned} \quad (5)$$

Where,

$C_1 = C_1(t)$ = The concentration of a chemical L at time t (kg / m^3)

$C_2 = C_2(t)$ = The concentration of a chemical M at time t (kg / m^3)

$C_3 = C_3(t)$ = The concentration of a chemical N at time t (kg / m^3)

k_1, k_2 = Rate constant (sec^{-1}), $k_1, k_2 > 0$

$C_1(0) = \beta =$ Initial concentration of a chemical L

$C_2(0) = 0 =$ Initial concentration of a chemical M

$C_3(0) = 0 =$ Initial concentration of a chemical N

with initial conditions,

$$C_1(0) = \beta, C_2(0) = 0, C_3(0) = 0 \quad (6)$$

System (5) can be written in the form

$$\begin{aligned} C_1' + k_1 C_1 &= 0 \\ -k_1 C_1 + k_2 C_2 + C_2' &= 0 \\ -k_2 C_2 + C_3' &= 0 \end{aligned}$$

Applying the Sawi transformation to the system of **Equation (5)**

$$\begin{aligned} S\{C_1' + k_1 C_1\} &= S\{0\} \\ S\{-k_1 C_1 + k_2 C_2 + C_2'\} &= S\{0\} \\ S\{-k_2 C_2 + C_3'\} &= S\{0\} \end{aligned} \quad (7)$$

By applying the linearity property of the Sawi transformation to **Equation (7)**, it is obtained

$$\begin{aligned} S\{C_1'\} + k_1 S\{C_1\} &= S\{0\} \\ -k_1 S\{C_1\} + k_2 S\{C_2\} + S\{C_2'\} &= S\{0\} \\ -k_2 S\{C_2\} + S\{C_3'\} &= S\{0\} \end{aligned} \quad (8)$$

Use the derivative function of the Sawi transformation in **Equation (8)**, where

$f'(t) = \frac{1}{\mu} T(C(t)) - \frac{1}{\mu^2} f(0)$, so that

$$\begin{aligned} \frac{1}{\mu} T(C_1) - \frac{1}{\mu^2} C_1(0) + k_1 S\{C_1\} &= 0 \\ -k_1 S\{C_1\} + k_2 S\{C_2\} + \frac{1}{\mu} T(C_2) - \frac{1}{\mu^2} C_2(0) &= 0 \\ -k_2 S\{C_2\} + \frac{1}{\mu} T(C_3) - \frac{1}{\mu^2} C_3(0) &= 0 \end{aligned} \quad (9)$$

Suppose $T(C_1) = S\{C_1\}$, $T(C_2) = S\{C_2\}$ and $T(C_3) = S\{C_3\}$, then the system of **Equation (10)** can be written as

$$\begin{aligned} \frac{1}{\mu} S\{C_1\} - \frac{1}{\mu^2} C_1(0) + k_1 S\{C_1\} &= 0 \\ -k_1 S\{C_1\} + k_2 S\{C_2\} + \frac{1}{\mu} S\{C_2\} - \frac{1}{\mu^2} C_2(0) &= 0 \\ -k_2 S\{C_2\} + \frac{1}{\mu} S\{C_3\} - \frac{1}{\mu^2} C_3(0) &= 0 \end{aligned} \quad (10)$$

Simplify the form of the system of **Equation (10)** and substitute the initial conditions (6) into the system of **Equation (10)**

$$\begin{aligned}
 \left(\frac{1}{\mu} + k_1\right) S\{C_1\} &= \frac{\beta}{\mu^2} \\
 -k_1 S\{C_1\} + \left(\frac{1}{\mu} + k_2\right) S\{C_2\} &= 0 \\
 -k_2 S\{C_2\} + \frac{1}{\mu} S\{C_3\} &= 0
 \end{aligned}
 \tag{11}$$

The system of **Equation (10)** can be written in the form $A\Phi = B$

$$\begin{bmatrix} \left(\frac{1}{\mu} + k_1\right) & 0 & 0 \\ -k_1 & \left(\frac{1}{\mu} + k_2\right) & 0 \\ 0 & -k_2 & \frac{1}{\mu} \end{bmatrix} \begin{bmatrix} S\{C_1\} \\ S\{C_2\} \\ S\{C_3\} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\mu^2} \\ 0 \\ 0 \end{bmatrix}
 \tag{12}$$

where

$$A = \begin{bmatrix} \left(\frac{1}{\mu} + k_1\right) & 0 & 0 \\ -k_1 & \left(\frac{1}{\mu} + k_2\right) & 0 \\ 0 & -k_2 & \frac{1}{\mu} \end{bmatrix}, \quad \Phi = \begin{bmatrix} S\{C_1\} \\ S\{C_2\} \\ S\{C_3\} \end{bmatrix}, \quad \text{and } B = \begin{bmatrix} \frac{\beta}{\mu^2} \\ 0 \\ 0 \end{bmatrix}$$

Solve the **Equation (9)** obtained by Cramer's method to get $S\{C_1\}$, $S\{C_2\}$, and $S\{C_3\}$ solutions.

$$S\{C_1\} = \frac{\text{Det}(A_1)}{\text{Det}(A)} = \frac{\beta}{\mu^2 \left(\frac{1}{\mu} + k_1\right)} = \frac{\beta}{\mu(1 + \mu k_1)}
 \tag{13}$$

$$\begin{aligned}
 S\{C_2\} &= \frac{\text{Det}(A_2)}{\text{Det}(A)} \\
 &= \frac{1}{\mu} \left(\frac{\beta \mu k_1}{(1 + \mu k_1)(1 + \mu k_2)} \right)
 \end{aligned}
 \tag{14}$$

$\frac{\beta \mu k_1}{(1 + \mu k_1)(1 + \mu k_2)}$ can be decomposed into a partial fraction by using partial fraction decomposition, so

Equation (14) becomes

$$\begin{aligned}
 S\{C_2\} &= \frac{1}{\mu} \left(\frac{\beta \mu k_1}{(1 + \mu k_1)(1 + \mu k_2)} \right) \\
 &= \left[\frac{\beta k_1}{(k_2 - k_1)} \right] \left(\frac{1}{\mu(1 + \mu k_1)} - \frac{1}{\mu(1 + \mu k_2)} \right)
 \end{aligned}
 \tag{15}$$

$$\begin{aligned}
 S\{C_3\} &= \frac{\text{Det}(A_3)}{\text{Det}(A)} \\
 &= \frac{\beta k_1 k_2}{\mu} \left(\frac{\mu^2}{(1 + \mu k_1)(1 + \mu k_2)} \right)
 \end{aligned} \tag{16}$$

By using partial fraction decomposition and polynomial long division, $\frac{\mu^2}{(1 + \mu k_1)(1 + \mu k_2)}$ can be decomposed into partial fractions, so **Equation (16)** becomes

$$\begin{aligned}
 S\{C_3\} &= \frac{\beta k_1 k_2}{\mu} \left(\frac{\mu^2}{(1 + \mu k_1)(1 + \mu k_2)} \right) \\
 &= \frac{\beta}{\mu} - \left[\frac{k_2}{(k_2 - k_1)} \right] \frac{\beta}{\mu(1 + \mu k_1)} + \left[\frac{k_1}{(k_2 - k_1)} \right] \frac{\beta}{\mu(1 + \mu k_2)}
 \end{aligned} \tag{17}$$

With the inverse Sawi transformation and its properties applied to **Equation (13)**, **Equation (15)**, and **Equation (17)**, the solutions C_1, C_2 , and C_3 obtained, respectively

$$\begin{aligned}
 C_1 &= S^{-1} \left\{ \frac{\beta}{\mu(1 + \mu k_1)} \right\} \\
 &= \beta S^{-1} \left\{ \frac{1}{\mu(1 + \mu k_1)} \right\} \\
 &= \beta e^{-k_1 t}
 \end{aligned} \tag{18}$$

$$\begin{aligned}
 C_2 &= S^{-1} \left\{ \left[\frac{\beta k_1}{(k_2 - k_1)} \right] \left(\frac{1}{\mu(1 + \mu k_1)} - \frac{1}{\mu(1 + \mu k_2)} \right) \right\} \\
 &= \left[\frac{\beta k_1}{(k_2 - k_1)} \right] S^{-1} \left\{ \left(\frac{1}{\mu(1 + \mu k_1)} - \frac{1}{\mu(1 + \mu k_2)} \right) \right\} \\
 &= \left[\frac{\beta k_1}{(k_2 - k_1)} \right] \left(S^{-1} \left\{ \frac{1}{\mu(1 + \mu k_1)} \right\} - S^{-1} \left\{ \frac{1}{\mu(1 + \mu k_2)} \right\} \right) \\
 &= \left[\frac{\beta k_1}{(k_2 - k_1)} \right] (e^{-k_1 t} - e^{-k_2 t})
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 C_3 &= S^{-1} \left\{ \frac{\beta}{\mu} - \left[\frac{k_2}{(k_2 - k_1)} \right] \frac{\beta}{\mu(1 + \mu k_1)} + \left[\frac{k_1}{(k_2 - k_1)} \right] \frac{\beta}{\mu(1 + \mu k_2)} \right\} \\
 &= S^{-1} \left\{ \frac{\beta}{\mu} \right\} - S^{-1} \left\{ \left[\frac{k_2}{(k_2 - k_1)} \right] \frac{\beta}{\mu(1 + \mu k_1)} \right\} + S^{-1} \left\{ \left[\frac{k_1}{(k_2 - k_1)} \right] \frac{\beta}{\mu(1 + \mu k_2)} \right\} \\
 &= \beta \left(S^{-1} \left\{ \frac{1}{\mu} \right\} - \left[\frac{k_2}{(k_2 - k_1)} \right] S^{-1} \left\{ \frac{1}{\mu(1 + \mu k_1)} \right\} + \left[\frac{k_1}{(k_2 - k_1)} \right] S^{-1} \left\{ \frac{1}{\mu(1 + \mu k_2)} \right\} \right) \\
 &= \beta \left(1 - \left[\frac{k_2}{(k_2 - k_1)} \right] e^{-k_1 t} + \left[\frac{k_1}{(k_2 - k_1)} \right] e^{-k_2 t} \right)
 \end{aligned} \tag{20}$$

Equation (18), **Equation (19)**, and **Equation (20)** are analytical solutions of the system of **Equation (5)** with initial conditions (6). **Equation (18)**, **Equation (19)**, and **Equation (20)** give the concentrations of

$C_1, C_2,$ and C_3 required for a chemical reaction, respectively. The concentration values of $C_1, C_2,$ and C_3 depend on time t and for different variations of $\beta, k_1,$ and k_2 . For the $C_1(t)$ concentration of a chemical substance L , the variation in value determines as follows.

$$\beta = 1(\text{kg} / \text{m}^3) \text{ and } k_1 = 0,5(\text{sec}^{-1})$$

$$\beta = 1(\text{kg} / \text{m}^3) \text{ and } k_1 = 1(\text{sec}^{-1})$$

$$\beta = 2(\text{kg} / \text{m}^3) \text{ and } k_1 = 0.5(\text{sec}^{-1})$$

$$\beta = 2(\text{kg} / \text{m}^3) \text{ dan } k_1 = 1(\text{sec}^{-1})$$

The predetermined combination of β and k_1 values, with the help of Mathematica, gets the value of the $C_1(t)$ concentration of a chemical L . The value of the $C_1(t)$ concentration of a chemical substance L for all variations in β and k_1 values that determine can see in **Table 2** below.

Table 2. $C_1(t)$ concentration of a chemical substance L during time t with different Value variations of β and

t (sec)	k_1			
	$\beta = 1(\text{kg} / \text{m}^3);$ $k_1 = 0.5(\text{sec}^{-1})$	$\beta = 1(\text{kg} / \text{m}^3);$ $k_1 = 1(\text{sec}^{-1})$	$\beta = 2(\text{kg} / \text{m}^3);$ $k_1 = 0.5(\text{sec}^{-1})$	$\beta = 2(\text{kg} / \text{m}^3);$ $k_1 = 1(\text{sec}^{-1})$
0	1	1	2	2
1	0.6065	0.3679	1.2131	0.7358
2	0.3679	0.1353	0.7358	0,2707
3	0.2231	0.0498	0.4463	0,0996
4	0.1353	0.0183	0.2707	0,0366
5	0.0821	0.0067	0.1642	0,0135
6	0.0498	0.0025	0.0996	0,0050

Table 2 shows that when the time t is from 0 to 6 seconds, the concentration of $C_1(t)$ of a chemical substance L continues to decrease for the four values of the combinations β and k_1 . It occurs because the concentration of $C_1(t)$ of a chemical substance L continues to decrease due to a chemical reaction that produces a new substance, namely a chemical substance M with a concentration of $C_2(t)$. The value of the $C_1(t)$ concentration of a chemical L decreases for a time t value from 1 to 6 seconds until the $C_1(t)$ concentration of a chemical L becomes 0 at time t . Further, **Table 2** shows that the greater β and k_1 ., the greater the concentration of $C_1(t)$ in a chemical substance L produced. The graph supports the results obtained in the table (see **Figure 2**).

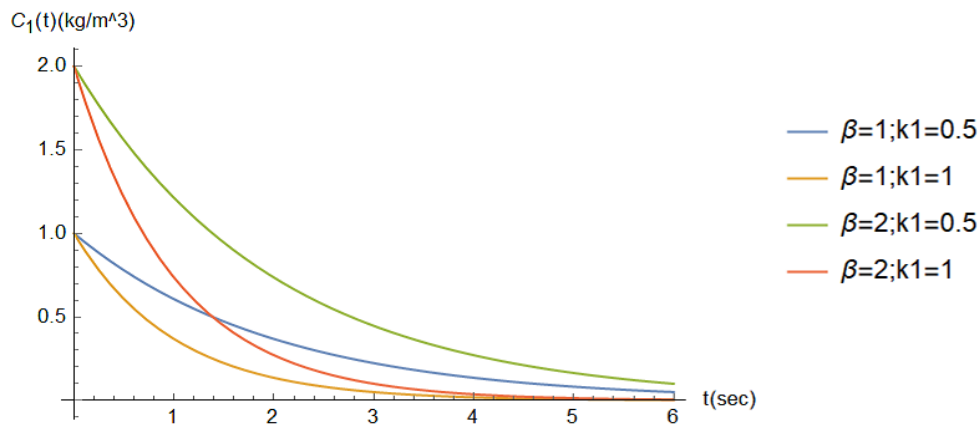


Figure 2. Graph of Solution $C_1(t)$, Which Depicts The Concentration of a chemical L during time t with a Variety Of Different Values β and k_1

Figure 2 shows that the $C_1(t)$ concentration curve of a chemical L with conditions $\beta=1(\text{kg}/\text{m}^3)$ and $k_1=0.5(\text{sec}^{-1})$ intersect, and an intersection point exists between the two curves. The intersection point means that when $t = 1.38629$ sec, the $C_1(t)$ concentration of a chemical L with conditions $\beta=1(\text{kg}/\text{m}^3)$ and rate constant $k_1=0.5(\text{sec}^{-1})$ will have the same amount of concentration as the concentration of $C_1(t)$ of a chemical L with conditions $\beta=2(\text{kg}/\text{m}^3)$ and rate constant $k_1=1(\text{sec}^{-1})$.

Determining the variations importance of β, k_1 , and k_2 as follows obtained the value of the $C_2(t)$ concentration of a chemical substance M and the value of the $C_3(t)$ concentration of a chemical N .

$$\beta=1(\text{kg}/\text{m}^3), k_1=0.5(\text{sec}^{-1}), \text{ and } k_2=1.5(\text{sec}^{-1})$$

$$\beta=1(\text{kg}/\text{m}^3), k_1=1(\text{sec}^{-1}), \text{ and } k_2=2(\text{sec}^{-1})$$

$$\beta=2(\text{kg}/\text{m}^3), k_1=0.5(\text{sec}^{-1}), \text{ and } k_2=1.5(\text{sec}^{-1})$$

$$\beta=2(\text{kg}/\text{m}^3), k_1=1(\text{sec}^{-1}), \text{ and } k_2=2(\text{sec}^{-1})$$

Using *Mathematica*, variations in the values of β, k_1 , and k_2 determined previously obtained the value of the $C_2(t)$ concentration of a chemical substance M . The results obtained for the value of $C_2(t)$ concentration of a chemical substance M with all variations in β, k_1 , and k_2 values determined can see in **Table 3** for $\beta=1$ and **Table 4** for $\beta=2$.

Table 3. The Concentration of $C_2(t)$ of a Chemical Substance M During Time t with Various Values of $\beta=1, k_1$, and k_2

t (sec)	$\beta=1(\text{kg}/\text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=1(\text{kg}/\text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$	$\beta=1(\text{kg}/\text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=1(\text{kg}/\text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$
0	0	0	0	0
1	0.1917	0.1571	0.2895	0.2325
2	0.1590	0.1165	0.1711	0.1170
3	0.1060	0.0736	0.0774	0.0473
4	0.0664	0.0450	0.0317	0.0180
5	0.0408	0.0273	0.0124	0.0067
6	0.0248	0.0166	0.0047	0.0025

Table 4. The concentration of $C_2(t)$ of a chemical substance M during time t with Various Values of $\beta=2, k_1$, and k_2

t (sec)	$\beta=2(\text{kg}/\text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=2(\text{kg}/\text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$	$\beta=2(\text{kg}/\text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=2(\text{kg}/\text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$
0	0	0	0	0
1	0.3834	0.3141	0.5790	0.4651
2	0.3181	0,2330	0.3422	0.2340
3	0.2120	0.1471	0.1547	0.0946
4	0.1329	0.0900	0.0633	0.0360
5	0.0815	0.0547	0.0247	0.0134
6	0.0497	0.0332	0.0094	0.0049

Table 3 and **Table 4** illustrate that initially, the value of the $C_2(t)$ concentration of a chemical substance M increases but decreases with increasing time t from 0 to 6 seconds. The $C_2(t)$ concentration value of a chemical substance M decreases during time t . The $C_2(t)$ concentration of chemical M initially increases due to a reduced $C_1(t)$ concentration of chemical L . It decreases due to an increase in the $C_3(t)$ concentration of chemical N . Furthermore, **Table 3** and **Table 4** show that the greater the value of β, k_1 , and k_2 , the greater the concentration of $C_2(t)$ in the chemical substance M produced. The graph in Figure 3 illustrates the same results as **Table 3** and **Table 4**.

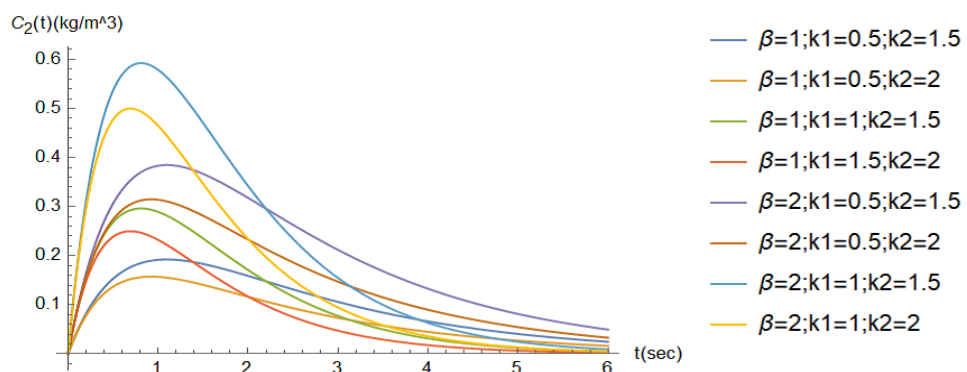


Figure 3. Graph of $C_2(t)$ solution, which depicts the concentration of a chemical M during time t with various values of β, k_1 , and k_2

Table 5 for $\beta=1$ and 6 for $\beta=2$ can see the results obtained for the $C_3(t)$ concentration value of a chemical N for all variations in β, k_1 , and k_2 values determined using *Mathematica*.

Table 5. The Concentration Of $C_3(t)$ of a Chemical Substance N During Time t With Various Values of $\beta=1, k_1$, and k_2

t (sec)	$\beta=1(\text{kg} / \text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=1(\text{kg} / \text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$	$\beta=1(\text{kg} / \text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=1(\text{kg} / \text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$
0	0	0	0	0
1	0.2018	0.2364	0.3426	0.3996
2	0.4731	0.5156	0.6936	0.7476
3	0.6709	0.7033	0.8729	0.9029
4	0.7982	0.8197	0.9500	0.9637
5	0.8771	0.8906	0.9809	0.9866
6	0.9254	0.9336	0.9928	0.9950

Table 6. The Concentration of $C_3(t)$ of a Chemical Substance N During Time t with Various Values of $\beta=2, k_1$, and k_2

t (sec)	$\beta=2(\text{kg} / \text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=2(\text{kg} / \text{m}^3);$ $k_1=0,5(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$	$\beta=2(\text{kg} / \text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=1,5(\text{sec}^{-1})$	$\beta=2(\text{kg} / \text{m}^3);$ $k_1=1(\text{sec}^{-1});$ $k_2=2(\text{sec}^{-1})$
0	0	0	0	0
1	0.4035	0.4728	0.6852	0.7992
2	0.9461	1.0312	1.3871	1.4953
3	1.3417	1.4066	1.7457	1.8058
4	1.5965	1.6393	1.9000	1.9274
5	1.7543	1.7811	1.9618	1.9731
6	1.8508	1.8672	1.9856	1.9901

Table 5 and **Table 6** illustrate that the value of the rate constant k_1 increases from 0.5 to 1 sec^{-1} , and the importance of the $C_3(t)$ concentration of a chemical N increases during time t . In addition, **Table 5** and **Table 6** also illustrate that the value of the rate constant k_2 increases from 1.5 to 2 sec^{-1} , and the value of the $C_3(t)$ concentration of a chemical N increases over time t . The $C_3(t)$ concentration of chemical N increases due to the reduced $C_2(t)$ concentration of chemical M when a chemical reaction occurs. Furthermore, **Table 5** and **Table 6** show that the greater β, k_1 , and k_2 , the greater the concentration of $C_3(t)$ in a chemical N produced. The graphical sketch in **Figure 3** presents the same results as in **Table 5** and **Table 6**.

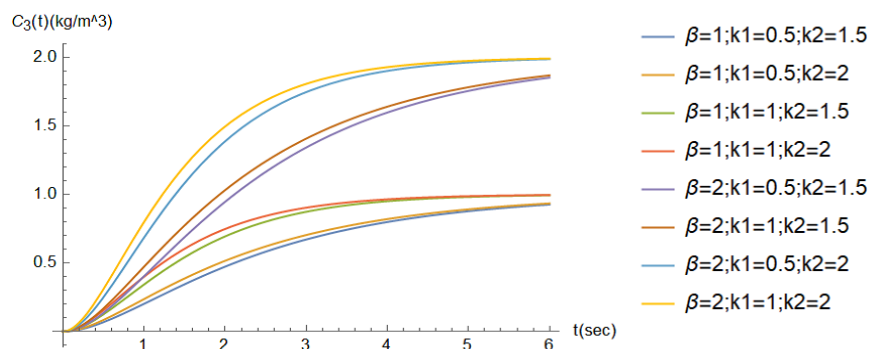


Figure 3. Graph of $C_3(t)$ Solution, Which Depicts The Concentration of A Chemical N During Time t With Various Values Of β, k_1 , and k_2

3.2 Numerical Example: Forced Motion Mass-Spring System

Consider a system of mass springs with two springs and two masses connected and bound at the right end and the horizontal plane conditions with different textures. Mass m_1 lies on a smooth plane, while m_2 lies on a rough plane so that when it moves, it produces a frictional force of $3\sin 2t$ Newton. $m_1 = m_2 = 1\text{ kg}$, $k_1 = 3\text{ N/m}$, and $k_2 = 2\text{ N/m}$. The position of the object x_1 when $t = 0$ is in its equilibrium position, while x_2 when $t = 0$ shifts to the right by 1 meter. For clarity, **Figure 4** depicts the mass-spring system [3].

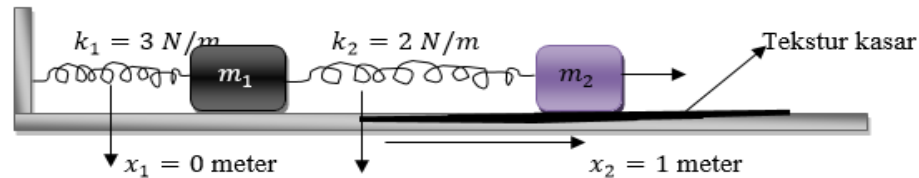


Figure 4. Mass-spring system with some grainy texture

Where can the mass-spring system in **Figure 1** be represented in the form of the following system of linear ordinary differential equations.

$$\begin{aligned} x_1''(t) + 5x_1(t) - 2x_2(t) &= 0 \\ -2x_1(t) + 2x_2(t) + x_2''(t) &= 3\sin 2t \end{aligned} \quad (21)$$

With initial conditions

$$\begin{aligned} x_1(0) = x_1'(0) &= 0 \\ x_2(0) = 1, x_2'(0) &= 0 \end{aligned} \quad (22)$$

Apply the Sawi transformation to the system of **Equation (21)**

$$\begin{aligned} S\{x_1''(t) + 5x_1(t) - 2x_2(t)\} &= S\{0\} \\ S\{-2x_1(t) + 2x_2(t) + x_2''(t)\} &= S\{3\sin 2t\} \end{aligned} \quad (23)$$

By applying the linearity property of the Sawi transformation to **Equation (23)**, it is obtained

$$\begin{aligned} S\{x_1''(t)\} + 5S\{x_1(t)\} - 2S\{x_2(t)\} &= S\{0\} \\ -2S\{x_1(t)\} + 2S\{x_2(t)\} + S\{x_2''(t)\} &= 3S\{\sin 2t\} \end{aligned} \quad (24)$$

Use the derivative function of the Sawi transformation in **Equation (24)**, where

$$f''(t) = \frac{1}{\mu^2} T(x(t)) - \frac{1}{\mu^3} f(0) - \frac{1}{\mu^2} f'(0),$$

so that

$$\begin{aligned} \frac{1}{\mu^2} S\{x_1(t)\} - \frac{1}{\mu^3} x_1(0) - \frac{1}{\mu^2} x_1'(0) + 5S\{x_1(t)\} - 2S\{x_2(t)\} &= 0 \\ -2S\{x_1(t)\} + 2S\{x_2(t)\} + \frac{1}{\mu^2} S\{x_2(t)\} - \frac{1}{\mu^3} x_2(0) - \frac{1}{\mu^2} x_2'(0) &= \frac{6}{(1+4\mu^2)} \end{aligned} \quad (25)$$

Suppose $T(x_1(t)) = S\{x_1(t)\}$ and $T(x_2(t)) = S\{x_2(t)\}$, then the system of **Equation (25)** can be written as

$$\begin{aligned} \frac{1}{\mu^2} S\{x_1(t)\} - \frac{1}{\mu^3} x_1(0) - \frac{1}{\mu^2} x_1'(0) + 5S\{x_1(t)\} - 2S\{x_2(t)\} &= 0 \\ -2S\{x_1(t)\} + 2S\{x_2(t)\} + \frac{1}{\mu^2} S\{x_2(t)\} - \frac{1}{\mu^3} x_2(0) - \frac{1}{\mu^2} x_2'(0) &= \frac{6}{(1+4\mu^2)} \end{aligned} \quad (26)$$

Simplify the form of the system of **Equation (26)** and substitute the initial conditions (22) into the system of **Equation (26)**

$$\begin{aligned} (1+5\mu^2)S\{x_1(t)\} - 2\mu^2 S\{x_2(t)\} &= 0 \\ -2\mu^3 S\{x_1(t)\} + (\mu+2\mu^3)S\{x_2(t)\} &= \frac{6\mu^3+4\mu^2+1}{1+4\mu^2} \end{aligned} \quad (27)$$

The system of **Equation (27)** written in the form $A\Phi = B$

$$\begin{bmatrix} (1+5\mu^2) & -2\mu^2 \\ -2\mu^3 & (\mu+2\mu^3) \end{bmatrix} \begin{bmatrix} S\{x_1(t)\} \\ S\{x_2(t)\} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{6\mu^3+4\mu^2+1}{1+4\mu^2} \end{bmatrix} \quad (28)$$

where $A = \begin{bmatrix} (1+5\mu^2) & -2\mu^2 \\ -2\mu^3 & (\mu+2\mu^3) \end{bmatrix}$, $\Phi = \begin{bmatrix} S\{x_1(t)\} \\ S\{x_2(t)\} \end{bmatrix}$, dan $B = \begin{bmatrix} 0 \\ \frac{6\mu^3+4\mu^2+1}{1+4\mu^2} \end{bmatrix}$.

Solve the **Equation (28)** obtained by Cramer's method to get $S\{x_1(t)\}$ and $S\{x_2(t)\}$ solutions.

$$\begin{aligned} S\{x_1(t)\} &= \frac{\text{Det}(A_1)}{\text{Det}(A)} \\ &= \frac{1}{\mu} \left(\frac{12\mu^5+8\mu^4+2\mu^2}{(1+\mu^2)(1+4\mu^2)(1+6\mu^2)} \right) \end{aligned} \quad (29)$$

The form $\frac{12\mu^5+8\mu^4+2\mu^2}{(1+\mu^2)(1+4\mu^2)(1+6\mu^2)}$ in **Equation (29)** can be decomposed using partial fraction decomposition, , so the partial fraction form written in **Equation (29)**, which has been obtained as follows

$$\begin{aligned} S\{x_1(t)\} &= \frac{1}{\mu} \left(\frac{12\mu^5+8\mu^4+2\mu^2}{(1+\mu^2)(1+4\mu^2)(1+6\mu^2)} \right) \\ &= \frac{4}{5(1+\mu^2)} + \frac{2}{5\mu(1+\mu^2)} + \frac{6}{5(1+6\mu^2)} - \frac{2}{5\mu(1+6\mu^2)} - \frac{2}{(1+4\mu^2)} \end{aligned} \quad (30)$$

$$\begin{aligned} S\{x_2(t)\} &= \frac{\text{Det}(A_2)}{\text{Det}(A)} \\ &= \frac{1}{\mu} \left(\frac{30\mu^5+20\mu^4+6\mu^3+9\mu^2+1}{(1+\mu^2)(1+6\mu^2)(1+4\mu^2)} \right) \end{aligned} \quad (31)$$

The form $\frac{30\mu^5+20\mu^4+6\mu^3+9\mu^2+1}{(1+\mu^2)(1+6\mu^2)(1+4\mu^2)}$ in **Equation (31)** can be decomposed using partial fraction decomposition, , so the partial fraction form written in **Equation (31)**, which has been obtained as follows

$$\begin{aligned}
 S\{x_2(t)\} &= \frac{1}{\mu} \left(\frac{30\mu^5 + 20\mu^4 + 6\mu^3 + 9\mu^2 + 1}{(1+\mu^2)(1+6\mu^2)(1+4\mu^2)} \right) \\
 &= \frac{8}{5(1+\mu^2)} + \frac{4}{5\mu(1+\mu^2)} - \frac{3}{5(1+6\mu^2)} + \frac{1}{5\mu(1+6\mu^2)} - \frac{1}{(1+4\mu^2)}
 \end{aligned} \tag{32}$$

With the inverse Sawi transformation and its properties applied to **Equation (21)** and **Equation (22)**, the solutions $x_1(t)$ and $x_2(t)$ obtained, respectively.

$$\begin{aligned}
 x_1(t) &= S^{-1} \left\{ \frac{4}{5(1+\mu^2)} + \frac{2}{5\mu(1+\mu^2)} + \frac{6}{5(1+6\mu^2)} - \frac{2}{5\mu(1+6\mu^2)} - \frac{2}{(1+4\mu^2)} \right\} \\
 &= \frac{4}{5} S^{-1} \left\{ \frac{1}{(1+\mu^2)} \right\} + \frac{2}{5} S^{-1} \left\{ \frac{1}{\mu(1+\mu^2)} \right\} + \frac{1}{5} S^{-1} \left\{ \frac{\sqrt{6}}{(1+6\mu^2)} \right\} - \frac{2}{5} S^{-1} \left\{ \frac{1}{\mu(1+6\mu^2)} \right\} \\
 &\quad - S^{-1} \left\{ \frac{2}{(1+4\mu^2)} \right\} \\
 &= \frac{4}{5} \sin t + \frac{2}{5} \cos t + \frac{\sqrt{6}}{5} \sin \sqrt{6}t - \frac{2}{5} \cos \sqrt{6}t - \sin 2t
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 x_2(t) &= S^{-1} \left\{ \frac{4}{5(1+\mu^2)} + \frac{2}{5\mu(1+\mu^2)} + \frac{6}{5(1+6\mu^2)} - \frac{2}{5\mu(1+6\mu^2)} - \frac{2}{(1+4\mu^2)} \right\} \\
 &= \frac{8}{5} S^{-1} \left\{ \frac{1}{(1+\mu^2)} \right\} + \frac{4}{5} S^{-1} \left\{ \frac{1}{\mu(1+\mu^2)} \right\} - \frac{1}{10} S^{-1} \left\{ \frac{\sqrt{6}}{(1+6\mu^2)} \right\} + \frac{1}{5} S^{-1} \left\{ \frac{\sqrt{6}}{\mu(1+6\mu^2)} \right\} \\
 &\quad - S^{-1} \left\{ \frac{\frac{1}{2}}{(1+4\mu^2)} \right\} \\
 &= \frac{8}{5} \sin t + \frac{4}{5} \cos t - \frac{\sqrt{6}}{10} \sin \sqrt{6}t + \frac{1}{5} \cos \sqrt{6}t - \frac{1}{2} \sin 2t
 \end{aligned} \tag{34}$$

Equation (33) and **Equation (34)** are analytical solutions of the system of **Equation (21)** with initial conditions (22). **Equation (33)** and **Equation (34)** are the displacements of the springs $x_1(t)$ and $x_2(t)$ for time t . By using *Mathematica*, it can be determined the value of $x_1(t)$ and $x_2(t)$ in time t . The importance of $x_1(t)$ and $x_2(t)$ can be seen in **Table 7** below.

Table 7. Displacement of Springs $x_1(t)$ and $x_2(t)$ During Time t

t (sec)	$x_1(t)$	$x_2(t)$
0	0	1
1	0.6006	1.0137
2	0.7622	1.7782
3	0.2312	-0.5440
4	-1.6624	-2.3254
5	-0.6431	-0.7686
6	1.3246	0.2756
7	-0.5963	1.3754
8	1.0602	1.5910
9	1.0891	0.1196
10	-2.2969	-1.6916

Table 7 shows that the displacement of the springs $x_1(t)$ and $x_2(t)$ at $t=0$ to $t=10$ changes with time t . Springs $x_1(t)$ and $x_2(t)$ move right first, then springs $x_1(t)$ and $x_2(t)$ move directly and left in time t , both in the same direction and in the antipodes. Until the spring system stops when the spring velocity slows down with time t . The graph in **Figure 5** illustrates the displacement of springs $x_1(t)$ and $x_2(t)$ over time t from $t=0$ to $t=10$.

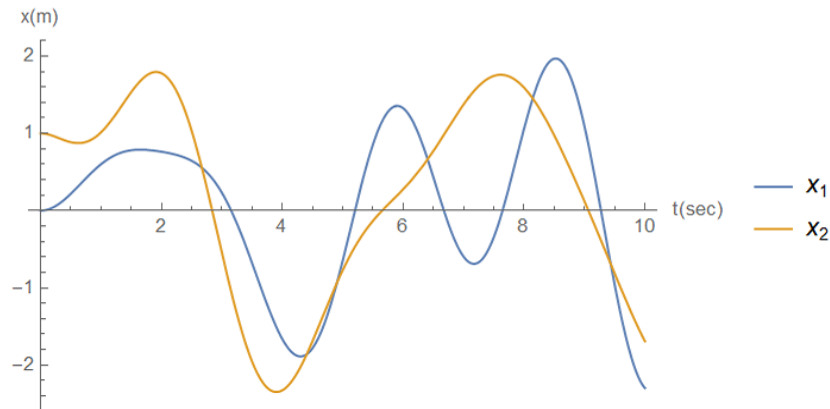


Figure 5. The Graphs Of The Solutions $x_1(t)$ and $x_2(t)$ Depict The Horizontal Displacement Of The Spring Associated With Masses m_1 And m_2 During Time t

4. CONCLUSIONS

We have shown that the Sawi transformation can solve a system of linear ordinary differential equations accompanied by initial values (the initial value problem). To obtain the solution, we do two main steps. Firstly, it was changing a system of linear first-order differential equations into a system of linear first-order differential equations involving the Sawi operator. Secondly, using the linear identity property, the derivative of the Sawi transformation function, the Sawi inverse transformation, and the required initial value replacement will give the expected analytical solution.

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