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A DIFFERENTIABLE STRUCTURE ON A FINITE DIMENSIONAL REAL VECTOR SPACE AS A MANIFOLD

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ABSTRACT

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There are three conditions for a topological space to be said a topological manifold of dimension n : Hausdorff space, second-countable, and the existence of homeomorphism of a neighborhood of each point to an open subset of $\mathbb{R}^n$ or n-dimensional locally Euclidean. The differentiable structure is given if the intersection of two charts is an empty chart or its transition map is differentiable. In this article, we study a differentiable manifold on finite dimensional real vector spaces. The aim is to prove that any finite-dimensional vector space is a differentiable manifold. First of all, it is proved that a finite dimensional vector space is a topological manifold by constructing a norm as its topology. Given a metric which is induced by a norm. Two norms on a finite dimensional vector space are always equivalent and they are determine the same topology. Secondly, it is proved that the transition map in the finite dimensional vector space is differentiable. As conclusion, we have that any finite dimensional vector space with independent norm topology choice is a differentiable manifold. As a matter of discussion, it can be studied that the vector space of all linear operators of a finite dimensional vector space has a differentiable manifold structure as well.

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1. INTRODUCTION

A manifold is a key word which gives a notion of a Lie group. It is interesting to study structures of Lie groups in the term of manifolds because a Lie group is nothing but a differentiable manifold. Furthermore, it corresponds to a Lie algebra. As the examples, it is well known that the notions of smooth functions, smooth maps, tangent bundles, cotangent bundles, vector bundles, left-invariant vector fields, tensor fields, k-forms, and symplectic manifolds (see [1], [2], [3], [4], [5]) are important parts to study manifolds. Roughly speaking, a Lie group is a group which has a differentiable manifold structure whose a map $\tau : G \times G \ni (a, b) \mapsto ab^{-1} \in G$ is differentiable. By computing a tangent space of G at the identity element $e \in G$, we have a Lie algebra isomorphism among $\mathrm{Lie}(G)$, the tangent space T_eG, and a Lie algebra $\mathfrak{g}$. In the other words, $\mathrm{Lie}(G) = \mathfrak{g}$ and the tangent space T_eG are isomorphic as a vector space in which a vector space isomorphism is given by the evaluation map $\eta: \mathrm{Lie}(G) \ni X \mapsto X_e \in T_eG$. In this research, applying the evaluation map, we get that the space $\mathfrak{g}$ is a finite dimensional space. Therefore, the dimension of $\mathfrak{g}$ is equal to dimension of G. Finally, we can observe that both $\mathrm{Lie}(G) = \mathfrak{g}$ and T_eG have an algebraic structure as a finite dimensional vector space. That is why the notion of a vector space is very important in Lie groups and Lie algebras. Particularly, we shall focus on the real vector space with finite dimension as a manifold.

Corresponding to a Lie algebra, it is well known that a Lie algebra comes from a Lie group by construction of its tangent space at the identity (see in [3] and [6]). The Lie algebra is also a vector space with certain conditions. Moreover, a vector space can act as a carrier space in representation theory of Lie groups [7]. Let G be a group and V be a vector space. A representation of G on V is given by a linear map $\psi_g: V \ni v \mapsto \psi_g(v) \in V$. We again state that a vector space is very important in the case of representation theory of Lie groups and Lie algebras.

In summary, a k-dimensional topological manifold is a topological space in which it is a Hausdorff space, a second countable, and a locally Euclidean space of dimension k. Equipped with a maximum smooth atlas then we have the notion of a differentiable structure on the topological manifold. The notion of manifolds come in many researches (see for example in reduction theory [8], information of geometrical evaluation [9], application of liquid metal related to manifold [10], symplectic structure on affine Lie group [11] , Poisson Lie group [12], and contact Lie groups [13]). Therefore, we believe that the significance of research in manifolds is very important both in pure and applied mathematics including in non-mathematics research areas. Manifolds arise in many branches of researches. Thus, we see it as very necessary to investigate the structure of a finite dimensional real vector space as a differentiable manifold.

In contrast to the previous results, in this paper we restrict our study to the case of finite dimensional real vector space. We investigate that a finite dimensional real vector space has a differentiable manifold structure. The idea of proof come from [1] and some facts about topology in a finite dimensional vector space. As a differentiable manifold, a real vector space of finite dimension has two structures as a topological manifold and as a derivation or a smooth structure. Firstly, the proof of a finite dimensional vector space as a topological manifold considered by construction an isomorphism linear map which is given $\psi: V \ni v \mapsto [v]_{\mathfrak{B}} \in \mathbb{R}^n$ where V is the real vector space of dimension n whose basis $\mathfrak{B} = \{v_1, v_2, \dots, v_n\}$ and $[v]_{\mathfrak{B}}$ is a coordinate of vector $v \in V$. Another construction of an isomorphism linear map can be given by $\eta: \mathbb{R}^n \ni x \mapsto x^k v_k \in V$ with $x^k v_k$ is written in the Einsten summation for $k = 1,2,3, \dots, n$. Secondly, we prove V has a differentiable structure by determining a smooth/differentiable transition map of V.

2. RESEARCH METHODS

To explore a finite dimensional real vector space as a differentiable manifold, we shall prepare some basic concepts. They are notions of topological spaces, topological manifolds, smooth/differentiable manifolds, a metric induced by a norm. Furthermore, on a finite dimensional vector space we have that both norms are equivalent and determine the same topology.

Definition 1 [1]. Let G be a topological space. The space G is said to be a topological manifold of dimension k if the following conditions are satisfied :

1. For each $g_1, g_2 \in G$ with $g_1 \neq g_2$, there exist open subsets $H_1, H_2 \subseteq G$ whose intersection is an empty set such that $g_1 \in H_1$ and $g_2 \in H_2$. We name G as a Hausdorff space.

2. The space G has a countable topological basis. We call G as a second countable.
3. Let g be an arbitrary element of G. There exists an open subset $H \subseteq G$ with $g \in H \subseteq G$ and an open subset $\widetilde{H} \subseteq \mathbb{R}^k$ such that the map defined by $\eta: H \ni g \mapsto \eta(g) \in \widetilde{H} \subseteq \mathbb{R}^k$ is a homeomorphism. In the other words, G is homeomorphic to $\mathbb{R}^k$ or k-dimensional locally Euclidean.

Example 1. The familiar examples of manifolds are Euclidean space $\mathbb{R}^n$, $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \subseteq \mathbb{R}^n$, the set of invertible matrices $\mathrm{GL}(n, \mathbb{R}^n)$.

Let $H \subseteq G$ be an open subset of a topological manifold of dimension k. A pair (H, η) is called a chart for G if U is homeomorphic to an open subset $\widetilde{H} = \eta(H) \subseteq \mathbb{R}^k$. In coordinate we write $\eta(g) = (x_1(g), x_2(g), \dots, x_k(g))$ in which x_i is a map defined from H into $\mathbb{R}$. In the other words, $(x_1, x_2, \dots, x_k)$ is the component function of η. We notice here that a chart for a topological manifold is not unique.

Definition 2 [1]. Let G be a k-dimensional topological manifold whose charts are (H_1, η_1) and (H_2, η_2). Let the intersection of these charts be not an empty set. The map defined by

$$\eta_1 \circ \eta_2^{-1}: \eta_2(H_1 \cap H_2) \to \eta_1(H_1 \cap H_2) \quad (1)$$

is called a transition map. The charts (H_1, η_1) and (H_2, η_2) are differentiable compatible each other if its intersection is an empty set or the transition map $\eta_1 \circ \eta_2^{-1}$ given by Equation 1 is a diffeomorphism.

In addition, let (H_α, η_α) charts for the k-dimensional topological manifolds G. An atlas $\mathfrak{H}$ for G is a collection of charts (H_α, η_α) such that $G = \bigcup_\alpha H_\alpha$. The atlas $\mathfrak{H}$ is differentiable if any two charts are differentiable compatible each other and the atlas $\mathfrak{H}$ is maximal if there is no larger atlas containing $\mathfrak{H}$. We mention here that some references use the notion ‘’smooth’’ instead of ‘’differentiable’’ notion.

Definition 3 [1]. A k-dimensional topological manifold G is said to be differentiable if it has a maximal differentiable atlas.

Based on **Definition 3** above, by construction a transition map on a k-dimensional topological manifold we can investigate whether an atlas has a differentiable structure or not. Furthermore, on a finite dimensional real vector space V we shall see that a chart can be determined as a linear map isomorphism between V and $\mathbb{R}^k$. The differentiable structure on V is derived from the differentiable structure of the transition map.

Definition 4 [1]. Let X be a non-empty set with two metrics d_1 and d_2. Let these metrics define topologies Γ_1 and Γ_2 for X. The metrics d_1 and d_2 are said topologically equivalent if determine the same topology. In other words, the metrics d_1 and d_2 are said topologically equivalent if $\Gamma_1 = \Gamma_2$.

Let d be a metric in a topological space X. The subset $Y \subseteq X$ is open in X if for each $y \in Y$, there exists $r > 0$ such that $y \in \mathfrak{B}_r(y) = \{x \in X \ ; d(x,y) < r\} \subseteq Y$. We denote by $\mathfrak{B}_r^1(y) = \{x \in X \ ; d_1(x,y) < r\}$ an open ball with respect to a metric d_1. Two metrics d_1 and d_2 are topologically equivalent if for all $x \in X$ and for all $r > 0$ there exist $r_1, r_2 > 0$ such that

$$B_{r_1}^1(x) \subseteq B_r^2(x) \quad \text{and } B_{r_2}^2(x) \subseteq B_r^1(x). \quad (2)$$

For a metric induced by a norm, then we have for simpler case in **Definition 4** written as follows.

Theorem 1 [1]. *Let V be a real finite dimensional vector space equipped with norms $[\![\circ]\!]_1$ and $[\![\circ]\!]_2$ which induce the metrics d_1 and d_2 for topologies Γ_1 and Γ_2 of V. The necessary and sufficient conditions for norms $[\![\circ]\!]_1$ and $[\![\circ]\!]_2$ to be topologically equivalent are the existence of $K_1, K_2 \in \mathbb{R}_{>0}$ such that $K_1[\![\circ]\!]_1 \leq [\![\circ]\!]_2 \leq K_2[\![\circ]\!]_1$.*

Proof. Firstly, we prove the necessary condition. Let $[\![\circ]\!]_1$ and $[\![\circ]\!]_2$ be topologically equivalent. It means that $\Gamma_1 = \Gamma_2$, we shall prove that there exist $K_1, K_2 \in \mathbb{R}_{>0}$ such that $K_1[\![\circ]\!]_1 \leq [\![\circ]\!]_2 \leq K_2[\![\circ]\!]_1$. It is enough to prove $K_1[\![p]\!]_1 \leq [\![p]\!]_2 \leq K_2[\![p]\!]_1$ for each $p \in B_1^1(0) \cap B_1^2(0) = \{y_1 \in V \ ; [\![y]\!]_1 < 1\} \cap \{y_2 \in V \ ; [\![y]\!]_2 < 1\}$. Another case can be generalized by multiple of p. By hypothesis we have **Equation (2)** for case $r = 1$ and $0 < r_1, r_2 < 1$. Namely, we have

$$B_{r_1}^1(0) \subseteq B_1^2(0) \quad \text{and } B_{r_2}^2(0) \subseteq B_1^1(0). \quad (3)$$

From **Equation (3)** then we have

$$[\![p]\!]_2 \leq \frac{[\![p]\!]_1}{r_1} \text{ and } [\![p]\!]_1 \leq \frac{[\![p]\!]_2}{r_2}. \tag{4}$$

Re-arranging **Equation (4)**, then we obtain the following in-equality :

$$r_2[\![p]\!]_1 \leq [\![p]\!]_2 \leq \frac{1}{r_1}[\![p]\!]_1. \tag{5}$$

Choose $K_1 = r_2$ and $K_2 = \frac{1}{r_1}$ then we complete the proof as desired.

Secondly, we assume that there exist $K_1, K_2 \in \mathbb{R}_{>0}$ such that for each $p \in V$, $K_1[\![p]\!]_1 \leq [\![p]\!]_2 \leq K_2[\![p]\!]_1$.

Let $p = p_1 - p_2$, then we have

$$K_1[\![p_1 - p_2]\!]_1 \leq [\![p_1 - p_2]\!]_2 \leq K_2[\![p_1 - p_2]\!]_{1.} \tag{6}$$

Equivalently, by writing **Equation (6)** in the following formula :

$$K_1 d_1(p_1, p_2) \leq d_2(p_1 - p_2) \leq K_2 d_1(p_1, p_2). \tag{7}$$

In the other words, we can choose $r_1 = r/K_2$ and $r_2 = K_1 r$ such that we obtain

$$B^1_{r/K_2}(p) \subseteq B^2_r(p) \text{ and } B^2_{K_1 r}(p) \subseteq B^1_r(p), \tag{8}$$

For all $p \in V$ and $r > 0$. Then we complete all proof.

■

Our conclusion state as a FACT 1 : Equivalent norms always give the same topology.

Theorem 2 [1]. *Let V be a real finite dimensional vector space equipped with two norms $[\![\circ]\!]_1$ and $[\![\circ]\!]_2$ which induce the metrics d_1 and d_2. Then norms $[\![\circ]\!]_1$ and $[\![\circ]\!]_2$ are equivalent or considered the same topology for V.*

Our conclusion state as a FACT 2 : Any two norms on a real finite dimensional vector space are equivalent.

3. RESULTS AND DISCUSSION

In this section, firstly we state some previous results corresponding to the Euclidean space $\mathbb{R}^n$ as a topological manifold. We give roughly summary from [9] and [14]. Secondly, we give the main result that a finite dimensional vector spase is a differentiable manifold.

Theorem 3 [15]. *Let V be a metric space whose metric is d. Then (V, d) is Hausdorff space.*

Proof. Let v_1 and v_2 be elements of V, $v_1 \neq v_2$, with $d(v_1, v_2) = r$. We consider two open balls $B_{\frac{r}{2}}(v_1) = \{p \in V \ ; d(v_1, p) < \frac{r}{2}\}$ and $B_{\frac{r}{2}}(v_2) = \{q \in V \ ; d(v_2, q) < \frac{r}{2}\}$. These imply that $v_1 \in B_{\frac{r}{2}}(v_1)$ and $v_2 \in B_{\frac{r}{2}}(v_2)$. The next step, we shall show that $B_{\frac{r}{2}}(v_1) \cap B_{\frac{r}{2}}(v_2) = \emptyset$. Using contradiction, let $v \in B_{\frac{r}{2}}(v_1) \cap B_{\frac{r}{2}}(v_2)$. Then we have $d(v_1, v) < \frac{r}{2}$ and $d(v_2, v) < \frac{r}{2}$. The property of metric implies that

$$d(v_1, v_2) \leq d(v_1, v) + d(v, v_2) < \frac{r}{2} + \frac{r}{2} = r.$$

This means that $d(v_1, v_2) < r$. The latter is false since $d(v_1, v_2) = r$. Thus, $B_{\frac{r}{2}}(v_1) \cap B_{\frac{r}{2}}(v_2) = \emptyset$.

■

Remark 1. The Euclidean space $\mathbb{R}^n$ and any subspace of $\mathbb{R}^n$ are metric spaces. Therefore, $\mathbb{R}^n$ is Hausdorff.

Theorem 4 [14]. *The Euclidean space $\mathbb{R}^n$ has countable basis.*

Proof. We claim that the set $\mathfrak{B} = \{B_r(x) \ ; \ x \in \mathbb{Q}, \ r \in \mathbb{Q}_{>0}\}$ of all open balls collection with rational centers x and rational radii $r > 0$ is a basis form $\mathbb{R}^n$. We recall that the set $\mathfrak{B}$ is a basis for $\mathbb{R}^n$ if each open set in $\mathbb{R}^n$ is a union of sets in $\mathfrak{B}$. In the other words, given an open set $V \subseteq \mathbb{R}^n$ and an element $v \in V$, then we must prove that there exists an open set $W \in \mathfrak{B}$ such that $p \in W \subseteq V$.

Let V be an open set in $\mathbb{R}^n$ and $v \in V$. Since U open, then there exists an open ball $B_r(v)$ with radius r is rational such that $v \in B_r(v) \subseteq U$. Let u be a rational element in $B_{\frac{r}{2}}(v)$. We shall show that $v \in B_{\frac{r}{2}}(u) \subseteq B_r(v) \subseteq U$. Since $u \in B_{\frac{r}{2}}(v)$ then $|u - v| < r/2$. Thus, $v \in B_{\frac{r}{2}}(u)$. Let $w \in B_{\frac{r}{2}}(u)$ then

$$|v - w| = |(v - u) + (u - w)| \leq |v - u| + |u - w| < \frac{r}{2} + \frac{r}{2} = r.$$

We obtain $w \in B_r(v)$. Therefore, the set $\mathfrak{B} = \{B_r(x)\, ;\, x \in \mathbb{Q},\ r \in \mathbb{Q}_{>0}\}$ is a basis for $\mathbb{R}^n$. But, the set $\mathbb{Q}$ and $\mathbb{Q}_{>0}$ are countable. Thus $\mathfrak{B}$ is a countable basis for $\mathbb{R}^n$.

■

Secondly, as mentioned before, we obtained that a real vector space of finite dimension equipped by a norm determines a differentiable manifold. An obtained metric is induced by the norm. Formally, in this section we state the main result in the following the Proposition 1. The result was obtained in ([1], pp.17—18) but we give the complete proof in the own way.

Proposition 1 [1]. *Let V be a real vector space of dimension n whose basis is $\mathfrak{A} = \{\varepsilon_k\}_{k=1}^n$. Let $[\![\cdot]\!]$ be a norm on V which induces a metric d. Then V is a topological manifold of dimension n. Moreover, define an isomorphism $\psi: \mathbb{R}^n \ni \alpha \mapsto \alpha^k \varepsilon_k \in V$. In this case $\alpha^k \varepsilon_k$ is an abbreviation for $\sum_{k=1}^n \alpha^k \varepsilon_k$ by applying the Einstein summation convention. Using a chart (V, ψ^{-1}) then V has a differentiable structure. Therefore, a space V is an n-differentiable manifold.*

Proof. Firstly, let V be a real vector space of dimension n whose basis is $\mathfrak{B} = \{\varepsilon_k\}_{k=1}^n$. Let $[\![\cdot]\!]$ be a norm on V which induces a metric d. Since V be a real vector space of finite dimension, then using FACT 2 we have that two norms on V are always equivalent. Moreover, using FACT 1 then these norms define the same topology on V. Furthermore, with respect to the norm $[\![\cdot]\!]$, a subset $U \subseteq V$ is an open set if for each $v \in V$ there exists $r > 0$ s.t. $B_r(v) = \{y \in V\ ;\ [\![y - x]\!] < r\} \subseteq U$. Let

$$\psi : \mathbb{R}^n \ni x = (x^1, x^2, \ldots, x^n) \mapsto \psi(x) = x^k \varepsilon_k \in V \tag{9}$$

be a continuos linear map. To see ψ is a linear map, let $a, b \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then $\psi(a + b) = (a^k + b^k)\varepsilon_k = a^k \varepsilon_k + b^k \varepsilon_k = \psi(a) + \psi(b)$. In addition, $\psi(\alpha a) = (\alpha a^k)\varepsilon_k = \alpha(a^k \varepsilon_k) = \alpha\psi(a)$. In the other hand, let $\psi(a) = 0$, then $a^k \varepsilon_k = 0$. But $\mathfrak{B} = \{\varepsilon_k\}_{k=1}^n$ is linear independence. This implies that $a^k = 0$ for each $k = 1,2,..,n$. Thus, $a = 0$. Therefore, ψ is one-one map. To prove ψ is onto. Let $y = a^k \varepsilon_k \in V$, then we can choose $a = [y]_{\mathfrak{B}} = (a^1, a^2, \ldots, a^n)$ such that $\psi(a) = y$. Thus ψ is onto. We proved that ψ is isomorphism linear map and homeomorphism (the invers map is given in the second proof). Therefore, topologically, Hausdorf space, second countable and Locally Euclidean space of dimension n are inhereted from $\mathbb{R}^n$. Thus, V is topological manifold.

Secondly, Construct the inverse map of ψ as written in **Equation (9)** in the as following form :

$$\psi^{-1}: V \ni v \mapsto [v]_{\mathfrak{B}} = (x^1, x^2, \ldots., x^n) \in \mathbb{R}^n \tag{10}$$

with $[v]_{\mathfrak{B}}$ denotes a coordinate of vector $v \in V$ with respect to the basis $\mathfrak{B}$. As proved before, ψ^{-1} is isomorphism linear map and homeomorphism as well. Since V is open set, then the pair (V, ψ^{-1}) defines a chart for V. Now, let $\mathfrak{B}' = \{\varepsilon'_k\}_{k=1}^n$ be another basis for V. Then we can find a transisition matrix from $\mathfrak{B}'$ to $\mathfrak{B}$ which is given by

$$A = ([\varepsilon'_1]_{\mathfrak{B}}\ \ [\varepsilon'_2]_{\mathfrak{B}}\ \ \ldots\ \ [\varepsilon'_n]_{\mathfrak{B}}) \tag{11}$$

which is invertible matrix. Of course, we also obtain $B = A^{-1}$ as transition matrix from the basis $\mathfrak{B}$ to the basis $\mathfrak{B}'$. For each $v \in V$, we have $[v]_{\mathfrak{B}} = A[v]_{\mathfrak{B}}'$. The corresponding isomorphism of this basis can be written as $\psi'(x) = x^l\, \varepsilon'_l$. From the transition matrix $A = (A)_{kl}$ we can consider the componens function ψ. Namely, $\psi_k = A_{kl}\, \varepsilon'_l$ with runs for all k. The transition map of charts is given as follows :

$$(\psi')^{-1} \circ \psi(x) = x'. \tag{12}$$

In this case $x' = ((x^1)', (x_2)', \ldots, (x_n)')$ is considered $(x^l)' \varepsilon'_l = x^k \varepsilon_k = A_{kl} \varepsilon^k e_l{}'$.
In other words, we have that $(x^l)' = A_{kl} x^k$ for each k. Therefore, the transition map in Equation 12 is diffeomorphism. Thus, V has differentiable structure and V is an n-differentiable manifold.

■

Corollary 1. *The space* $\mathrm{M}_{m\times n}(\mathbb{R})$ *of all* $m \times n$ *real matrices is a differentiable manifold of dimension* mn.

Proof. We can observe that under usual matrix addition and scalar multiplication, the set $\mathrm{M}_{m\times n}(\mathbb{R})$ is a finite dimensional real vector space. Indeed, using Proposition 1, we have $\mathrm{M}_{m\times n}(\mathbb{R})$ is an mn-differentiable maniflod. We can see that $\mathrm{M}_{m\times n}(\mathbb{R})$ can be identified by $\mathbb{R}^{mn}$. Therefore, the differentiable structure is determined by the atlas of the chart $\mathrm{Id}_{\mathbb{R}^{\mathrm{mn}}}$.

■

4. CONCLUSIONS

A real vector space of finite dimension equipped with a norm in which induces a metric has structure as a differentiable manifold. The choice of norms is independent since in finite dimensional vector space any two norms define the same topology. In the other words, if V is a real vector space of dimension n whose basis is $\mathfrak{A} = \{\varepsilon_k\}_{k=1}^{n}$ in which it is equipped with the norm $[\![\cdot]\!]$ inducing a metric d, then V is a topological manifold of dimension n. Moreover, by defining an isomorphism $\psi: \mathbb{R}^n \ni \alpha \mapsto \sum_{k=1}^{n} \alpha^k \varepsilon_k \in V$, we applied a chart (V, ψ^{-1}) to prove that V has a differentiable structure. For further research, it is interesting to study a Lie algebra constructed from left-invariant vector fields of a Lie group.

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