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PROMOTING MATHEMATICAL JUSTIFICATION THROUGH REALISTIC MATHEMATICS EDUCATION CLASSROOM

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Abstract

The importance of mathematical justification in all classrooms is emphasized by reform movements in mathematics education. However, previous studies reveal that encouraging justification in mathematics classes is troblesome for mathematics teachers at all school levels. This present study is aimed to find out how justification help students to develop their mathematical understandings. To answer this question, we first determined the role of justification in teaching and learning process and how good students in justifying their answers. Design research was chosen as an appropriate mean to achive that goal. The study was conducted in a state university in Yogyakarta involving four students from fifty-one students as our focus group and one of researchers as a lecturer. Justification was the focus on this study and to promote it we chose RME as an approach that served as a learning environment. In this study, all data collected from classroom observation, group observation, students' works, video recordings, field notes during teaching experiment, and students' final writen test. Students' works then analyzed using a justification rubric and describe thoroughly by considering other data collected during teaching learning process. The study revealed that there were some roles of justification in RME classroom, namely encouraged conceptual understanding, supported mathematical skill, promoted long lasting skill, and influenced social relationship. In this study, most students in our focus group have already made a good justification which was assessed by the points in the CLEAR rubric. Students wrote their solutions equipped by signs or simbols, made labels, used a complete sentence to answer the question, and elaborated procedures throroughly.

Keywords: justification, RME, design research, mathematical communication

1. Introduction

Learning to argue about mathematical ideas is a fundamental part in understanding mathematics and think mathematically. Through communicating, students learn to explore, analyze, and enhance their understanding of mathematical relationships and reasoning. Conner et al. (Conner et al., 2017) stated justification as a way of communicating to other in which we thoroughly present ideas or claims along with the reasons behind them. Justification defines as arguments demonstrating truth from a claim using agreed statements and mathematical forms of reasonings (Sarumaha, 2018). Through their social interactions with teachers and other students, students learn what constitutes a legitimate argument.

Justification is also a practice in learning mathematics (Melhuish et al., 2015). It is a main element of the generalization process (Liang, 2020). Justification is a disciplined process that serves various functions, including verification of arguments, clarification or supplies of knowledge into a result or phenomena, and organizing information (Bell & 1976, 1976; Cioe et al., 2015; De Villiers, 2002; Hähkiöniemi et al., 2022; Mata-pereira & Ponte, 2017). Each field of study dicipline (and society) has its own criterias for establishing a hypothesis or theory as a (working) truth, as well as its own standards for what constitutes justification (Staples et al., 2012). In contrast to mathematics, where a new finding must be supported by rigorous logical arguments that proves the truthfulness of mathematical claims, or a proof, in science, scientific theories have been formed based on observations. Justification, in a mathematics classroom, is a process of persuading others that a claim is true or correct. Justification, regardless of the exact term used, is crucial to performing and understanding mathematics and ought to be taught at all grade levels (NCTM, 2000).

For more students, developing and analyzing arguments to justify or refute a generalization is a difficult feat (Lannin et al., 2006). Providing justifications that are compelling to oneself and others is necessary when justifying a proposition or an outcome (Mata-pereira & Ponte, 2017). Here, the teacher needs to take control in teaching and learning process. Previous studies reveal that encouraging justification in mathematics classes is troblesome for mathematics teachers at all school levels (Francisco, 2022). Teacher should guide discussion by having students explain and justify their ideas both vocally and in writing. The teacher must create an environment in the classroom where students can learn to ask one another for justifications and clarifications in order to enable them to understand why they startegies or claims apply in varied circumtances (NCTM, 1991). This type of environment requires negotiating norms about what constitutes a valid justification (Cioe et al., 2015; Francisco, 2022; McCrone, 2005; Wood et al., 2006). In their study, James et al. (James, Philiben, et al., 2016) also demonstrated that students' ability to reason, explain, and justify themselves may be improved in classrooms where they have the opportunity to engage in mathematical argumentation and justification.

For many teachers and students, however, implementing these practices in mathematics classrooms is difficult, especially as they might not have previously encountered settings where justification of mathematical reasoning plays a prominent role (Hunter & Anthony, 2008). Nowadays, a sizable and expanding amount of research on argumentation, justification, and proof in mathematics education is reported from many countries. However, there was still lack of focus in what kind of learning environment that could promote students' justification. To help teachers, we need a learning approach in which students are encouraged to develop their justification as well as mathematical understanding. This approach is vastly known as Realistic Mathematics Education (RME). RME is a didactic approach or a domain-specific instruction theory for mathematicsfounded in Netherland (Van Den Heuvel- Panhuizen & Drijvers, 2014). In Indonesia, this approach was adapted and implemented known as *Pendidikan Matematika Realistik Indonesia* (PMRI).

As one of countinuously reconceptualizing approaches, RME's principles were reworded from time to time. Van den Heuvel-Panhuizen (Van Zanten & Van Den Heuvel-Panhuizen, 2021) divided these principles into six parts, namely the activity, the relaity, the level, the intertwinement, the interactivity, and the guidance principle. In this present study, we follow these principles. The idea of mathematics as a human activity (Freudenthal & Reidel, 1983) becomes the focal point of the activity principle. Students are encouraged to gain mathematical skills and insights by being treated as active participants in the learning process. According to the reality principle (Van Den Heuvel-Panhuizen, 2000), learning mathematics begins by mathematizing reality, which progresses from events with rich, relevant context to mathematical concepts (Van Den Heuvel-Panhuizen & Drijvers, 2014). In learning mathematics, students follow different levels of comprehension, according to the level principle. It begins with the use of informal context-related solutions, progresses through a variety of shortcuts and schematizations, and ends with the ability to apply formal procedures (Van Zanten & Van Den Heuvel-Panhuizen, 2021). Learning components are blended during instruction according to the intertwinement principle (Sarumaha et al., 2018; Van Den Heuvel-Panhuizen & Drijvers, 2014). According to the interactivity principle, learning mathematics is both social and a personal activity. The guidance principle emphasizes how crucial it is to give students the chance to reinvent mathematics under the guidance of teachers.

Justification underlies in the interactivity and guidance principle of RME. However, the teaching and learning process follows all the principles stated. We wish to establish the classroom norm where additional explanation of how students' findings relate to and represent the problem situation is required during social interactions or class discussions. Students become aware of the need for more information through discussions in the classroom about the consequences of giving a rule without reason. Students can be encouraged to learn how to get advanced type of justification by developing social and sociomathematical norms in the classroom that compel students to justify using combination of verbal, numerical and graphical strategies. (Kazemi, 1998; Rizkianto, 2013). We want to promote mathematical justification in the classroom in which students construct valid mathematical explanation for their work. In this present study, the term justification is employed to describe an argument that proves the truth of a claim or student's response, which is used to support accepted claims and mathematical forms of argument. The process of justification involves having students express their thinking and how they determine something to be true or accurate. According to Boaler & Staples's research (Boaler & Staples, 2008), encouraging students to use justification could help varied groups who are heterogeneously organized achieve more equal outcome.

Another reason why we implementing RME approach in this present study was because in RME, the teachers have a prominent part in facilitating the learning process by giving the students a setting that allows them to deepen their understanding of mathematics (Van den Heuvel-Panhuizen, 2020; Van Zanten & Van Den Heuvel-Panhuizen, 2021). In other words, we attempt to promote students' mathematical justification in RME classroom environment. Based on previous studies about justification conducted in some countries, we realize that those studies still lacked attention in learning environment where students can be encouraged to develop their ability to justify their answers. Because of that, we propose a reserach question, namely how can justification help students to develop their mathematical understanding? To answer this research question, we divided it into two sub research questions, (1) what is (are) the role(s) of justification in learning mathematics? and (2) How good students can justify their answers?

2. Method

Design research was the research method utilized in this present study to address the research question and accomplish the study's objectives. A design experiment is carried out in three stages: planning the experiment, conducting the experiment in the classroom, and the retrospective analysis (Gravemeijer & Cobb, 2006; Sarumaha et al., 2018). In the preparing phase, we designed a sequence of instructional activities containing conjecture of students' answers. The conjecture of Hypothetical Learning Trajectory (HLT) could be ajusted based on students' learning during the experiment. Teaching experiment phase consisted of two cycles which were divided into six lessons. Data collection of the teaching experiment were all analyzed, and the findings in the restrospective analysis were utilized to respond to the research question, reach conclusions, and redesign the HLT. This article will only elaborate the result of first cycle of the experiment. Retrospective analysis result was used to form conclusions and provide the answer to the research issue.

This present study involved 51 students in a state university in Yogyakarta who took logic and set course where one of the researchers acted as a lecturer in that course (we refer a lecturer as a teacher in this entire article). The first cycle of the study was conducted from September to October 2022. This cycle was divided into six lessons and one final test. The focus of the study was a small group consisted of four students who were chosen randomly from the whole class.

Before conducting the experiment, we did classroom observation and preparation for the lessons that would be taught. Analyzing data started from classroom observation, group observation, students' works, video recordings, field notes during teaching experiment, and students' final writen test. Students' written test graded using a rubric which was derived from Cioe et al (Cioe et al., 2015).

Table 1. CLEAR Rubric for assessing students' responses

Acronym	Score Point	0	1
C	Calculations	No work is shown.	Solutions show mathematical ideas involved.
		Some work is missing.	Answer includes procedures and/or simbols or tables, or graphs, or pictures.
L	Labels	No. labels are included.	Solutions are correctly labeled.
		Items are incorrectly labeled.	
E	Evidence	Solutions do not support the decision made.	Solutions support the decision made.
		Evidence is missing for some part of the problem.	Evidence is provided for all parts of the problem,
A	Answer the question	Answer is inaccurate.	Answers the question asked using a complete sentence (capitalization and punctuation)
		Answer does not address the question being asked.	Answer is accurate.
R	Reasons why	The procedure is not supported by mathematical evidence, or no explanation is provided.	Procedure is identified.
		The response shows confusion about content ideas and concepts.	Procedure is explained and what it means.
			Clear understanding is shown of content ideas and concepts.

Table 1 showed score point and explanation for each acronym to assess how good students' justification was. Rubric CLEAR was adapted and used in this present study to check and compare students' works based on the criterias given. What students have attained during the lessons were showed by their written test results where they have to solve some problems. Their works then begraded according to the score point in the rubric and by the acronym stated in the table 1. Since one of our study questions on how good students justify their answers, the discussion here maintained in the field of justification.

3. Result and Discussion

3.1 Result

Logical thinking in higher education entails generalizing rather than the focused computations of high school. It offers fascinating chances to evaluate the validity of extended arguments. In this present study, to encourage students to consider what constitutes a good justification for general proposition, we developed challenging mathematical problems and asked questions related to justification. This type of reasoning connects the mathematical activity in high school and the reasoning required in higher education. In this study, we helped students to use justification to develop rich understandings of mathematics logic. Focused was placed on how justififcation support students' understanding about mathematics logic.

In this present study, we used open-ended mathematical tasks or nonroutine problems to promote students' discussion and reasoning. Open-ended mathematical assignments offer excellent opportunity for students to participate in meaningful mathematical processes, which including conjecturing, generalizing, and justifying (James, Casas, et al., 2016). Students must create their own techniques, draw from several content areas of mathematics knowledge, and evaluate the accuracy of their answers when solving nonroutine issues (Hiebert et al., 1996). Providing them with mathematically meaningful tasks was one of them. Therefore, in the six lessons we conducted, the preparing phase of our design research was mostly spent with preparing and designing rich mathematical tasks. To help students constructed their own understanding about logic, activities were designed regarded to RME principles. In each lesson, we asked students worked in groups, discussed their strategies, their findings and in the end, we held classroom discussion where every group presented their results.

In the beginning of our study, we discovered that students found it difficult to provide justificaton for their responses. In other words, they did not seem to comprehend justification or lacked the skills to address questions like "please, justify your answer" or "how are you sure your answer is true?" Instead of addressing the questions properly by showing how they came up with the answer or why they think their answer was correct, most students explained their steps in solving the tasks or gave examples in which their answer worked. After discussing the result of the first and second lesson, we tried a different way to get a better answer from students and to be able to enhance students' justification. To help students develop their mathematical justification, we change our original questions into some casual words where students could grasp the ideas. Some questions that can be asked consistently to strive toward the goal of presenting mathematically valid justifications such as "what is changing in this situations?"; "what stays the same?"; "how does your finding relate to the problem situation?"; "how are you sure that that your answer will work for ... (some other statements)?; "Will your rule always work?"; and "how do you know your rule will always work?" These revised questions prompted students to focus on sense making and reasoning about relationship. By posing these queries and encouraging students to do the same with their friends, the norm for creating and assessing mathematical arguments was formed (Rizkianto, 2013).

Orchestrating discussion in every lesson was a must in our study. We found the way students convinced their friends about their strategies gradually changed along with the way we change our questions. We expected students to examine each other's justifications and analyze their reasoning during class discussion. To encourage this, we posed some questions, for example "how does your explanation relate to his or her explanation?"; "does his or her expalanations describe why his or her rule always works?"; "can you explain why your rule will or will not always work?"; and "how does this problem relate to other problems that we have done?". Thus, we anticipated that whole-class discussion would be a chance for students to review the veracity of other students' answers as well as a chance to exchange ideas.

After the sixth lesson, we conducted the final writen test for all students. However, what we discussed here will only be centered for the small group's results. The data from classroom observation, classroom discussion, small group discussion, video recordings, field notes and the final written test were collected and studied altogether to discernthe development of students' mathematicaljustification process.

According to the experiment result, we can classify the roles of jusification as follow

a. Encourage conceptual understanding

When learning about logic, students learned how to formulate a statement, to determine its truth value, to apply reasoning to each step that they took in solving problems, to produce new information from the statements given, and to reach a conclusion. In practice, students in the classroom started to explain their ideas or strategies to their friends, giving examples as they establish their claim as the correct answer. However, as our way to promote students to justify their answers in discussion, some students in our focus group started to question how his or her claim was true or how to make people believe that.

From the figure 1, question 1a (left), the student wrote his answer (right) by giving explanation why he didwhat he did. He showed his understanding about a prime number and summation of odd and even number. He elaborated his answer by giving reasons and why he thought the way he answered.

1. Tabel berikut adalah catatan eksplorasi yang telah dilakukan oleh Anya.

2	4	$4 = 2 + 2$
3	9	$9 = 2 + 7$
5	25	$25 = 2 + 23$

a. Berdasarkan kolom ketiga di tabel tersebut, tuliskan satu konjektur yang dapat dihasilkan oleh Anya.

b. Apakah konjektur yang kamu tuliskan di bagian a adalah suatu pernyataan?

c. Apa yang perlu kamu lakukan untuk menunjukkan apakah konjektur yang kamu tuliskan di bagian a benar atau salah?

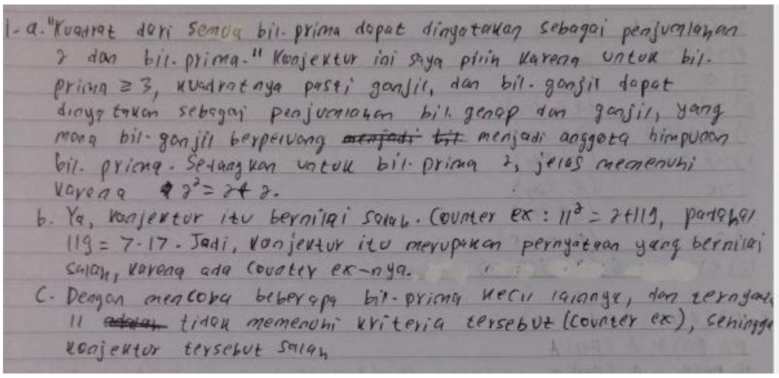
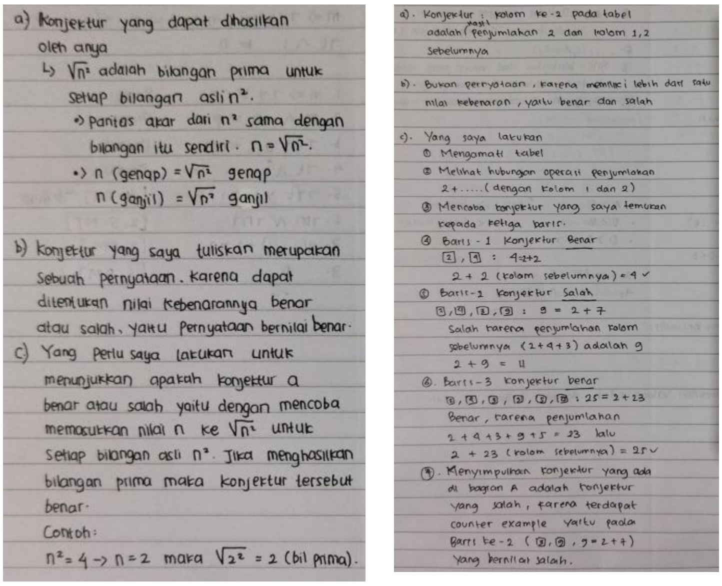

1. a. "Kuadrat dari semua bil. prima dapat dinyatakan sebagai penjumlahan 2 dan bil. prima." Konjektur ini saya pilih karena untuk bil. prima ≥ 3, kuadratnya pasti ganjil, dan bil. ganjil dapat dinyatakan sebagai penjumlahan bil. genap dan ganjil, yang mana bil. ganjil berpeluang ~~menjadi~~ ~~bil~~ menjadi anggota himpunan bil. prima. Sedangkan untuk bil. prima 2, jelas memenuhi karena $2^2 = 2 + 2$.
b. Ya, konjektur itu bernilai salah. Counter ex: $11^2 = 2 + 119$, padahal $119 = 7 \cdot 17$. Jadi, konjektur itu merupakan pernyataan yang bernilai salah, karena ada counter ex-nya.
c. Dengan mencoba beberapa bil. prima kecil lainnya, dan ternyata 11 ~~adalah~~ tidak memenuhi kriteria tersebut (counter ex), sehingga konjektur tersebut salah

Figure 1. Problem (left) and student result (right)

He also wrote a counter example to show that his conjecture was wrong, as he has already known that by given one counter example, the truth value of a conjecture which has been built would be void. It also demonstrated in his answer, part 1c, where he tried to apply some numbers for the conjecture, number eleven did not give a satisfy result.


a) Konjektur yang dapat dihasilkan oleh anya
↳ $\sqrt{n^2}$ adalah bilangan prima untuk setiap bilangan asli n^2.
•) Paritas akar dari n^2 sama dengan bilangan itu sendiri. $n = \sqrt{n^2}$.
•) n (genap) = $\sqrt{n^2}$ genap
n (ganjil) = $\sqrt{n^2}$ ganjil

b) Konjektur yang saya tuliskan merupakan sebuah pernyataan. Karena dapat ditentukan nilai kebenarannya benar atau salah. yaitu pernyataan bernilai benar.

c) Yang perlu saya lakukan untuk menunjukkan apakah konjektur a benar atau salah yaitu dengan mencoba memasukkan nilai n ke $\sqrt{n^2}$ untuk setiap bilangan asli n^2. Jika menghasilkan bilangan prima maka konjektur tersebut benar.
Contoh:
$n^2 = 4 \rightarrow n = 2$ maka $\sqrt{2^2} = 2$ (bil prima).

a). Konjektur : kolom ke-2 pada tabel adalah hasil penjumlahan 2 dan kolom 1,2 sebelumnya

b). Bukan pernyataan, karena memiliki lebih dari satu nilai kebenaran, yaitu benar dan salah

c). Yang saya lakukan
1 Mengamati tabel
2 Melihat hubungan operasi penjumlahan 2 + ... (dengan kolom 1 dan 2)
3 Mencoba konjektur yang saya temukan kepada ketiga baris.
4 Baris - 1 Konjektur Benar
$\boxed{2}$, $\boxed{4}$: 4 = 2 + 2
2 + 2 (kolom sebelumnya) = 4 ✓
5 Baris - 2 Konjektur Salah
$\boxed{3}$, $\boxed{4}$, $\boxed{2}$, $\boxed{9}$: 9 = 2 + 7
Salah karena penjumlahan kolom sebelumnya (2 + 4 + 3) adalah 9
2 + 9 = 11
6. Baris - 3 Konjektur benar
$\boxed{3}$, $\boxed{4}$, $\boxed{3}$, $\boxed{2}$, $\boxed{5}$, $\boxed{25}$: 25 = 2 + 23
Benar, karena penjumlahan 2 + 4 + 3 + 9 + 5 = 23 lalu 2 + 23 (kolom sebelumnya) = 25 ✓
7. Menyimpulkan konjektur yang ada di bagian A adalah konjektur yang salah, karena terdapat counter example yaitu pada Baris ke - 2 ($\boxed{3}$, $\boxed{9}$, 9 = 2 + 7) yang bernilai salah.

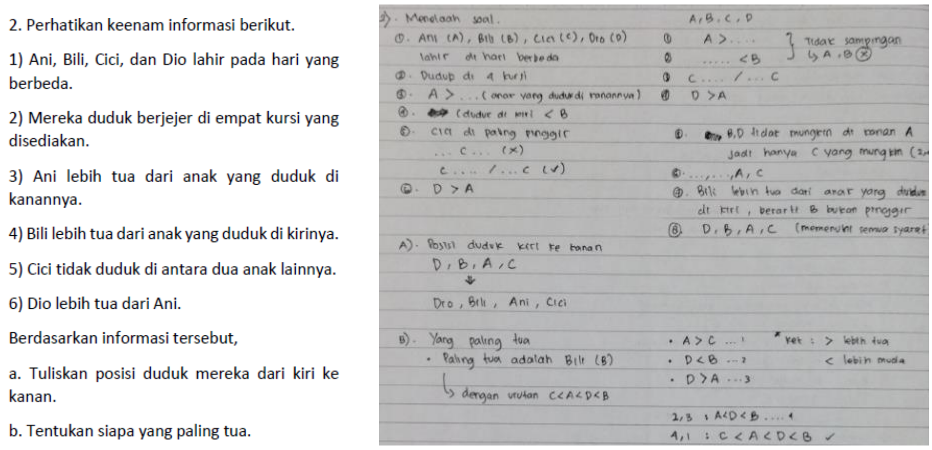
Figure 2. Students' answers (b) and (c).

From figure 2 (b and c), we see that those students stillwere not able to justify their answer and even theirclaims were incorrect. Students (b) tried to show her rule by stating why her rule worked. However, she did not give adequate reasons as how she cameup with the rule and why it worked for the problem. Meanwhile, another student (c) stated her answer without any supporting statements as why her answers were right. It also could be seen from their answers where they just make statements about the truth value of a conjecture, butwithout justifying them. For c, students wrote down their procedures to come up wth the answer by given some examples why they worked. They draw a conclusion from their examples sincethey thought those examples could be taken as a generalization of the problem.

We learnt from students answers in figure 2 that their conceptual understanding about a conjecture and proving its value depended on the truth value of some examples they proposed. By giving those examples to answer the problem, they draw a conclusion. It was still hard for students to justify their answers. Probably, it happened because we have not posed some supporting questions they needed to justify their thought thoroughly. They have not realized that a general statement has more mathematical potency than testing specific cases.

b. Support mathematical skill

Besides enhancing students conceptual understanding, mathematical justification also supports students matehmatical skill. Look at figure 3, on the left there is a problem while on the right is one of students' answers. Even though, we did not ask student to justify her answer, she came up with the solution by expaining how and why she ended up with it. Student in figure 3 seemed trying to solve the problem with more than one method. We assumed that she tried to be certained of the answer by reelaborating the problems, as could be seen from her left and right steps. The interesting thing is that in both ways, she made her steps countable by giving a number in each step she took. It might help her to come up with the solution and it also made teacher aware toward her effort in justifying her answer.


2. Perhatikan keenam informasi berikut.

1) Ani, Bili, Cici, dan Dio lahir pada hari yang berbeda.

2) Mereka duduk berjejer di empat kursi yang disediakan.

3) Ani lebih tua dari anak yang duduk di kanannya.

4) Bili lebih tua dari anak yang duduk di kirinya.

5) Cici tidak duduk di antara dua anak lainnya.

6) Dio lebih tua dari Ani.

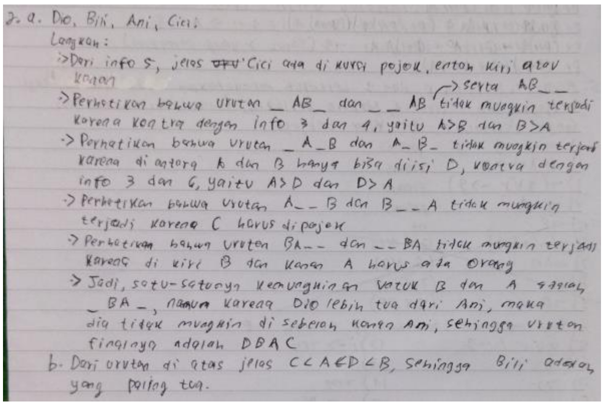
Berdasarkan informasi tersebut,

a. Tuliskan posisi duduk mereka dari kiri ke kanan.

b. Tentukan siapa yang paling tua.

Figure 3. Problem 2 and student's answer (a).

Figure 4 is another example of student's answer for this second problem. Here, students provided his justification by explaining clearly what he did in each step to find the solution. The similarity we saw from these two figures (figure 3 and figure 4) is that both students wrote down their procedures systematically. They elaborated each of their steps and explained what they did at that time, one step, one action. Even though we also could see student's answer in figure 3, figure 4 seemed to be more precised and detailed in explaining.


2. a. Dio, Bili, Ani, Cici

Langkah:

-> Dari info 5, jelas ~~dari~~ Cici ada di kursi pojok, entah kiri atau kanan

-> Perhatikan bahwa urutan _AB_ dan _ _AB serta AB_ _ tidak mungkin terjadi karena kontra dengan info 3 dan 4, yaitu A>B dan B>A

-> Perhatikan bahwa urutan _A_B dan A_B_ tidak mungkin terjadi karena di antara A dan B hanya bisa diisi D, namun dengan info 3 dan 6, yaitu A>D dan D>A

-> Perhatikan bahwa urutan A_ _B dan B_ _A tidak mungkin terjadi karena C harus di pojok

-> Perhatikan bahwa urutan BA_ _ dan _ _BA tidak mungkin terjadi karena di kiri B dan kanan A harus ada orang

-> Jadi, satu-satunya kemungkinan untuk B dan A adalah _BA_, namun karena Dio lebih tua dari Ani, maka dia tidak mungkin di sebelah kanan Ani, sehingga urutan finalnya adalah DBAC

b. Dari urutan di atas jelas C < A < D < B, sehingga Bili adalah yang paling tua.

Figure 4. Student's answer (b)

From those results, we know justification helps students to develop their communication and representational skill, such as simbols and to make connection with other representations. When they justify their previous knowledge and reasonings, students combined concepts or understanding of something new.

c. Promote long lasting skills

Students had the chance to strengthen their mathematical and social skills by learning mathematics in an RME classroom where socio-mathematical norms were in practice. Here, we saw how students develop their habits or characters through promoting justification in every lesson we conducted. We noted some changes in the teaching learning process gradually became better from before. Students were not only eager to explain their answers to their friends, but also curious in listening how others worked. They showed their mathematical understanding sometimes by providing representation such as simbol or drawings.

They intended to try make others believe what they thought was true or correct, and they also learnt other methods from other friends, other claims, or rules. Discussion on justification provide students the chance to see how a rule applies in various scenarios, make genaralization to similar circumtances, and consider their own logic for the usefullness of their rules. The skill we talked about here were critical thinking, independent, and sportive. The understanding of what supports and refuses a justification should help students develop justifications that are more formal as they progress in their education (Mata-pereira & Ponte, 2017).

d. Influence social relationship

As we know justification is a way of communication, we also believe it has influence in social relationship. The effects do not only influence how students work with their peers, but also how interaction among all students in the classroom manage to develop a socialmathematical norm. It also affects the role of teacher as the source of truth where anything always goes with what the teacher says (Sarumaha, 2016). Moreover, the popular student who titled to be the smartest or the best one will not be going to be the main person to solve any mathematical probems. Through justification, students reasoned with themselves, found out how they came up with a solution or an idea and the reasoning behind why they thought their thinking, or their claims were correct. Students became more independent, be themselves and believe in their ways. Gradually, students became able to solve any problems themselves and colaborated with their friends in sharing ideas or opinions.

To assess students' justification, we used a rubric. Table 1 is a CLEAR rubric (Cioe et al., 2015) we adapted and used in this present study. It aimed to assess whether students made a good justification or not.

a. Calculations (solutions)

We changed into solutions here since in learning logic we barely deal with complicated calculations or number. We found out that students have shown mathematical ideas in each of the answer given. It also inculded using simbols or sometimes figures (like signs) to represent their expalanation and how their claims were right.

b. Labels

All our focus group students have already been able to label their works based on their mathematical understanding. Labeling was one way to make they kept track of their works step by step. Even though they emerged with various and different labels, they gave elaboration and information in which the label represented their findings.

5. Perhatikan keempat informasi berikut.

1) Jika Maki berlatih, maka Yuji tidak berjaga atau Inu tidak istirahat.

2) Jika Maki dan Nobara tidak berlatih, maka Panda istirahat.

3) Tidak benar bahwa jika Yuji atau Yuta berjaga maka Gojo istirahat.

4) Yuta tidak berjaga dan Inu istirahat.

Tentukan satu informasi yang jika ditambahkan ke keempat informasi di atas dapat mengantarkan pada simpulan bahwa **Nobara berlatih.**

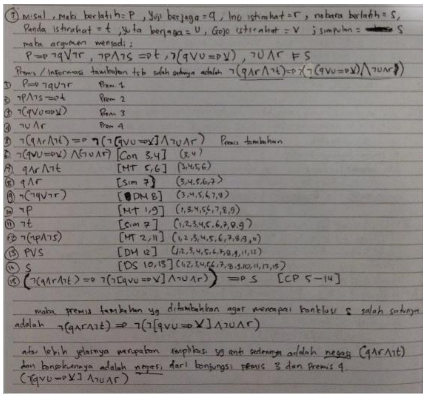


Figure 5. Problem 5 and student's answer (a)

Figure 5 and 6 presented how students made decision themselves in labeling their solution. In logic, we use small letters to represent a statement. From what we saw in the figures, students started by labeling the statements given in the problem their own ways.

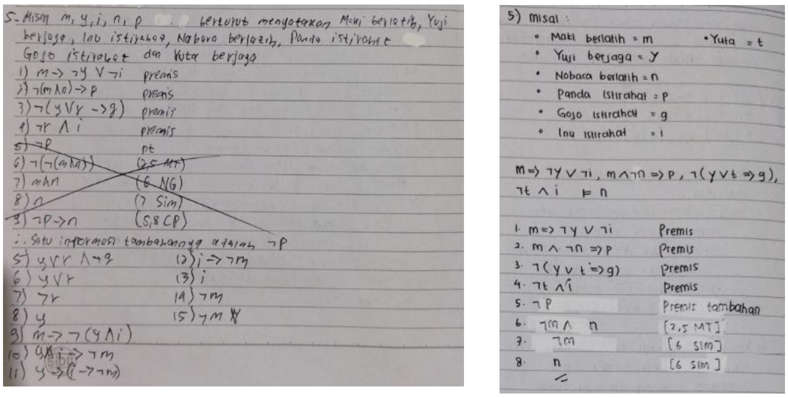


Figure 6. Students' answer (b) and (c).

c. Evidence

In justification, evidence is not enough by giving some examples in which the rule works. In this case, we proposed the idea of deductive justification. Using a deductive justification, students presented a general rationale for why the rule holds true in all situations of the scenario. Unlike the general examples, this justification does not specify a specific occurence. Rather, it defines a common relationship that is valid in all circumtances. Student's answer in figure 1 and figure 4 illustrated his ability to deductively justify his answer.

d. Answer the question

All students in the focus group have written down their answers even though not all of them got the correct ones. When students discussed their answers or strategies with others, they frequently did so without explaining why they thought such solutions or strategies applied to all situations. It is unsure just how this rule came or why it operates as it does. Thus, we classified this type of explanation as no justification or procedural justification. In another case, students extend his or her justification by examining several cases or giving examples, we called it as an empirical justification where students frequently claim that a generalization is accurate after simply testing a few examples (like student's answer in figure 2).

e. Reason why

Students have previously specified the features that make up his or her procedures which is then recognized, defined, and described. It also showed clear understanding about the ideas and concepts they provided. Look back to figure 5 and 6 students' answer, they provided and numbered each of their steps. It was not surprising since we also used these procedures in our lessons. However, since students wrote it down here, it means that they thought it as a useful way insolving problems. So, it can be said that students have already given the reason as to why they did what they did. Through all problems questioned inthe final test, almost all students gave the reasons using step by step procedures and explanations of those procedures but not all of them were qualifiedas a mathematical justification method.

3.2 Discussion

The focus of the current study was not just on how to apply the RME approach in classroom, but also – and perhaps more significantly – on students' mathematical justification. The need of justification here, comes up from the uncertainty of how a rule was derived or why a rule works. In the current study, we put our attention on justification because its importance as a disciplinary practice as well as its function as a learning practice (Staples et al., 2012). Students can improve their comprehenshion of mathematics and their mathematical skill by using justification as a learning practice. In other word, justification means to learn and to do mathematics.

Increasing students' mathematical reasoning requires getting them to justify, make hypotheses, and generalize (Widjaja, 2014). From students' works in the previous section, we can see that justification developed an understanding that allows students to build generalizations for related circumtances. Students generally considered the meaning they have built for their generalizations when they developed justification. They inquired about the nature of their rules or procedures were, for example why am I doing this.

Looking at the way students justified their answers, we can classify the reasoning within justification into some different levels. Students either openly stated the explanation for the claim (stated the explicit reasoning), described evidence in favor of the claim (the implicit reasoning), or created factual statements that teacher utilized to validate the claim (teacher used students' statements in reasoning). These results aligned with study conducted by Hähkiöniemi et all. (2022). In some cases, we found that students did justified their answers without any addition questions given. It happened probably because justification had already become a habit or daily practice in learning mathematics. Since the teacher always required students to justify their answers in teaching learning process, when it came to the written test, students also had the need to justify their answers.

For the result of our study, we also found that teacher plays a key role in scaffolding the exact questions and prompts that help students advance from presenting their solution techniques to justifying, defending, and generalising their solution strategies. It is basically what Hunter (Hunter, 2007) in his research found. Teacher can make assessment or evaluation from students' justification process. Justification can act as a window for teacher to see how students think about a problem, in which students stumble and have misconceptions. This information is useful to consider and redesign the next tasks or problems in the classroom. As stated by Mata-Pereira and Ponte (Mata-pereira & Ponte, 2017) a highly challenging action that aims to enhance students' mathematical reasoning may evoke a generalization or a justification, but it often requires numerous follow-up acts from the teacher.

Moreover, through justification in groups, students shared their opinions and approaches that they used to solving problems. What we saw here that students helped each other, sharing their understanding to others about the strategies they found. The result, of course, becomes richer since many strategies could be utilized. Justification also assists students establish the connection between generalization and practice. By structuring the task to focus on a sense making using a single example and frequently prompting for explanations,students provided better justifications and ultimately gained a deeper understanding of mathematics. Some students in our focus group also showed their ability to provided conter- examples. As stated by Meta-Pereira and Ponte (Mata-pereira & Ponte, 2017) that students shouldcooperate in justifying claims using mathematical concepts that they have already grasped or disputing claims by offering counter examples.

Futhermore, it is also essential that students improve their understandings about what validates a statement or a claim since refusing claims or statements laid on authority, perception, common sence, or special phenomena. Attending to and enhance students' justification may require a high-level skill set involving awareness of the nature of mathematical justification and the capacity to observe and make sense of students' mathematical reasoning and content-related techniques. To be able to produce better quality of written or oral justification, teacher needs to engage and provide students more frequently to work with justification tasks (Liang, 2020). The fundamental role of teacher in developing students' justification skill (Ayalon & Hershkowitz, 2017; Kazemi et al., 2021) depends on teacher's ability to perform negotiation of norms and responsiveness of classroom interactions. Teacher plays a crucial role in challenging students' thinking by asking them why and writing down what they are thinking on the board in orderfor students to focus on the important mathematicalconcepts (Widjaja, 2014).

The result in this present study yielded from a small-scale sample which consisted of four students. To understand more about justification, we recommend further study involving a large number of students. It also possible to lengthen the time or to add more meetings to see other aspect about how students' mathematical reasonings develop to promote better justification.

4. Conclusion

Based on the result and discussion stated above, we found some roles of justification in the RME classroom, namely encourage conceptual understanding, support mathematical skill, promote long lasting skill, and influence social relationship. Using rubric CLEAR to see how good students can justify their answer, we conclude that most of all students in our focus group have already been able to justify their answers. They showed mathematical ideas, including made some signs or pictures. They also included labels in their works with the explanation of those labels. One of them has already been able to make deductive justification by providing a general argument that clearly present a general rationale for why the rule holds true in all situations of the scenario. Most students answered using complete sentenses and the answers were accurate. Students also showed their procedures and explained what they meant. Their understandings were presented through content ideas and concepts.

To put it into a nutshell, mathematical justification did helped students develop their mathematical understandings. Through RME principles that we implemented in the lesson, students grew up in a learning environment where their ideas and strategies were valued equally. However, additional research is needed to confirm the results of this study due to the small number of participants involved.

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