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COMPARATIVE ANALYSIS OF NUMERICAL SOLUTIONS OF EULER, RK-4, ABM-4 AND RKCOM4 METHODS OF INITIAL VALUE PROBLEMS IN NONHOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract
Non-homogeneous second-order differential equations are often used in various mathematical models in physics, engineering, and system dynamics. Numerical solutions are the main alternative when analytical solutions are difficult to obtain. This study compares the performance of the Euler, Runge-Kutta 4th order (RK-4), Adams-Bashforth-Moulton 4th order (ABM-4), and Runge-Kutta Contra Harmonic Mean 4 (RKCoM4) numerical methods in solving initial value problems (MNAs) in non-homogeneous second-order differential equations. The analysis was carried out by comparing the numerical calculation results of each method using the Mean Absolute Error (MAE) method. The results of numerical calculations and simulations show that the RK-4 and ABM4 methods provide higher accuracy than the Euler and RKCoM4 methods for 2 cases of non-homogeneous second-order differential equations.

Keywords: *ABM4, differential equations, euler's method, numerical methods, RK4, RKCoM4.*

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1. INTRODUCTION

Differential equations are one of the fields in mathematics that are very important in various other fields of science, such as physics, engineering, biology, economics, and so on [9]. Non-homogeneous second-order linear differential equations often appear in many practical applications, including in fluid dynamics models, mechanical oscillations, and electrical circuits [10],[19]. In this context, solving initial value problems in such differential equations is often of particular interest because analytical solutions are not always easy to obtain or even non-existent. Therefore, numerical solutions are a common choice to solve these problems [9], [10], [14].

Numerical solution of differential equations requires an appropriate method to obtain accurate results. Some numerical methods that are often used include the Euler method [17], the Runge-Kutta method [4]-[6],[15]-[17], [41], and the Adams-Bashforth method [2], [14], [17], [22], [41]. Research on Euler numerical method has been very much done both in the form of its modification and its application in various fields of physics [16]-[19], [23], [27], [28], [31], [35]; Biomathematics or Applied Mathematics [1], [3]-[5], [7], [8], [20], [25], [29], [32]. In this paper, four numerical methods are studied, namely Euler's method, Runge-Kutta 4th order method, Adam Bashforth-Moulton (Predictor-Corrector) method and Runge-Kutta 4th order Conta-harmonic Mean method. These four methods have their own advantages and disadvantages in terms of accuracy, stability and computational efficiency. Thus, a comparison between the four methods is very important to determine the most effective method in solving non-homogeneous second-order linear differential equations. This study aims to compare the numerical solution of the initial value problem in non-homogeneous second-order linear differential equations using four numerical methods, namely the Euler method, the Runge-Kutta method of order 4, the Adams Bashforth-Moulton method 4 and the Runge-Kutta method of order 4 Contra-Harmonic Mean. Comparative analysis will be carried out based on the Mean Absolute Error (MAE) accuracy value of the error results of each method of numerical solution results against the exact solution. It is hoped that the results of this study can provide a deeper understanding of the performance of each numerical method in solving non-homogeneous second-order linear differential equations.

2.1 RESEARCH METHODS

This study aims to compare the numerical solution of the initial value problem in non-homogeneous second-order linear differential equations using four numerical methods, namely the Euler Method, Runge-Kutta 4th order, Adam Bashforth-Moulton 4th order, and Runge-Kutta 4th order Contra-Harmonic Mean.

2.1 Definition of Initial Value Problem (MNA)

The initial value problem (MNA) in differential equations is the problem of finding a solution to a differential equation that satisfies certain initial conditions. In general, the MNA for a first-order differential equation can be written as[38],[40]:

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

where:

$f(x, y)$ is the function given,

$y(x)$ is the solution sought,

x_0 is the starting point,

y_0 **is** the initial value of the function in x_0.

As for the nth order differential equation, MNA is defined as:

$$\boldsymbol{y^{(n)}(x) = f(x, y, y', \dots, y^{(n-1)}), \qquad y(x_0) = y_0, \qquad y'^{(x_0)} = y'_0, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}}$$

MNA is widely found in modeling various phenomena.

2.2. Non-Homogeneous Second-Order Linear Differential Equation

The most widely applied higher-order ordinary differential equation is the second-order differential equation. In this discussion, a first-degree non-homogeneous second-order PD will be used. At the initial stage, the second-order non-homogeneous linear differential equation with predetermined initial conditions will be formulated mathematically **[3]**, **[31]**. The general equation used in this study has the form:

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + p_2(x)\,y = R(x) \tag{2}$$

or,

$$y''(x) + p(x)\,y'(x) + q(x)y(x) = R(x)$$

with $R(x) \neq 0$, initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_{\prime 0}$. The chosen equation has been previously calculated for its exact solution so that it can be compared with the numerical solution.

2.2. Numerical Method

The four numerical methods that will be used to solve the differential equation are as follows:

a. Euler's method

This method is a simple numerical method, which is based on the approximation using the first derivative of the function**[18]**,**[22]**.

$$\begin{aligned} y(x_{r+1}) &= y(x_r) + h\,f(x_r\,, y_r) \;; \qquad r = 0, 1, 2, \cdots, h \\ y_{r+1} &= y_r + h\,f(x_r, y_r) \end{aligned} \tag{3}$$

b. Runge-Kutta Method Order 4 (RK4)

The 4th Order Runge-Kutta numerical method is a more complex numerical method than the Euler method, which provides higher accuracy. The Runge-Kutta method is derived from a modification of the Taylor Series, and is obtained by finding the four slopes of the line obtained from the function $f(x_i, y_i)$ of the differential equation. The gradients are named k_1, k_2, k_3 and k_4 respectively, with the following form of **Equation [17], [19], [39], [41]**:

$$\begin{aligned} k_1 &= h\,f(x_r, y_r) \\ k_2 &= h\,f\left(x_r + \frac{1}{2}h, y_r + \frac{1}{2}k_1\right) \\ k_3 &= h\,f\left(x_r + \frac{1}{2}h, y_r + \frac{1}{2}k_2\right) \\ k_4 &= h\,f(x_r + h,\ y_r + k_3) \\ y_{r+1} &= y_r + \frac{1}{6}(k_1 + 2\,k_2 + 2\,k_3 + k_4) \end{aligned} \tag{4}$$

c. Adam-Bashforth-Moulton 4-Step Method (ABM4)
The Adam-Bashforth-Moulton 4 method is a predictor-corrector method that uses the numerical solution of the previous point to estimate the next solution **[5], [17], [23]**. The form of the predictor and corrector equations is as follows:
Adams-Bashforth (AB4) Predictor

$$y_{i+1} = y_i + \frac{h}{24}\,[55\,f_i - 59\,f_{i-1} + 37\,f_{i-2} - 9\,f_{i-3}] \tag{5}$$

Adams-Moulton (AM4) Corrector

$$y_{i+1} = y_i + \frac{h}{24}\,[9\,f_{i+1} + 19\,f_i - 5\,f_{i-1} + \,f_{i-2}] \tag{6}$$

d. Runge-Kutta Contra Hamonic Mean Orde 4 (RKCoM4)
The fourth-order Runge-Kutta method has been widely modified such as the modification of the classical Runge-Kutta method based on arithmetic mean, centroidal mean, square root mean, geometric mean, harmonic mean, heronian mean, and counter-harmonic mean. RKCoM4 is a modified result of the fourth-order Runge-Kutta method, with the following equation form **[36],[37], [41]**:

$$\begin{aligned} k_1 &= f(t_n,\ y_n) \\ k_2 &= f(t_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}hk_1) \\ k_3 &= f\left(t_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{8}hk_1 + \tfrac{3}{8}hk_2\right) \\ k_4 &= f(t_n + h,\ y_n + \tfrac{1}{4}hk_1 - \tfrac{3}{4}hk_2 + \tfrac{3}{2}hk_3) \end{aligned} \tag{7}$$

$$y_{n+1} = y_n + \frac{h}{3}\left(\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_4^2}{k_3 + k_4}\right)$$

2.3. Accuracy Testing

After the numerical solution of the second-order differential equation is obtained using the four numerical methods, the next step is to compare the numerical solution with the exact solution of the calculated differential equation to obtain its accuracy. Accuracy testing is done by calculating the error between the numerical solution and the exact solution. The errors used are global errors and local errors at certain points. This method will give an idea of the accuracy of each numerical method. A common measure of prediction error used is the mean absolute error (MAE). Mean Absolute Error (MAE) is one of the methods used to measure the accuracy of a prediction model. The MAE value represents the average absolute error between the result and the actual value. In general, the form of the MAE equation is as follows**[13]**.

$$MAE = \frac{\sum |y_t - x_t|}{n} \tag{8}$$

where:
Error : Actual value of period *t* - Forecasted value of period *t*
y_t : Actual data or value of period *t*
x_t : The value of the forecasting result for period *t*

3. RESULTS AND DISCUSSION

In this section, numerical solutions to the initial value problem of non-homogeneous second-order differential equations will be discussed, using several numerical methods, namely the Euler method, the Runge-Kutta method of order 4 (RK4), the Adam Bashforth-Moulton 4-step method (ABM4), and the Runge-Kutta method of order 4 Contra-Harmonic Mean (RKCoM4).

3.1 Non-homogeneous 2nd Order Differential Equation

In this section, two examples of non-homogeneous second-order differential equations (PDs) and their initial values will be given, which have been calculated and the general and exact solutions determined. The non-homogeneous PD used as a case example in this discussion is a PD whose homogeneous part is in the form of trigonometric functions, exponential functions and quadratic functions. This non-homogeneous 2nd order differential equation is used to analyze how the best numerical solution of the initial value problem of non-homogeneous 2nd order PD using the Euler method, the Runge-Kutta 4th order (RK4) method, the Adam Bashforth-Moulton 4-step (ABM4) method, and the Runge-Kutta 4th order Contra-Harmonic Mean (RKCoM4) method.

a. Case 1 for non-homogeneous 2nd order Differential Equations

The first case discussed is the second-order non-homogeneous differential equation in the form of trigonometric and exponential equations, as follows:

$$y'' - 7y' + 12y = 8\sin x + e^{2x}\ , \qquad y(0) = -1\ ,\ \ y'(0) = 1,\ \ 0 \le x \le 1$$

After solving by using one of the ways to solve the Non Homogeneous 2nd Order PD, the general solution of the PD is obtained, as follows:

$$y = c_1\, e^{3x} + c_2\, e^{4x} + \frac{28}{85}\cos x + \frac{44}{85}\sin x + \frac{1}{2}e^{2x}$$

The general solution is solved by using the initial value $y(0) = -1$, and $y'(0) = 1$ to find the value of c_1 and c_2 so that the exact solution value is obtained as follows

$$y = \frac{-578}{85}\ e^{3x} + \frac{845}{170}\ e^{4x} + \frac{28}{85}\cos x + \frac{44}{85}\sin x + \frac{1}{2}e^{2x}$$

b. Case 2 for non-homogeneous 2nd order Differential Equations:

$$y'' - 5y' + 6y = t^2\ , \qquad y(0) = 1\ ,\ \ y'(0) = -2,\ \ 0 \le x \le 1$$

After solving by using one of the ways to solve the Non Homogeneous 2nd Order PD, the general solution of the PD is obtained, as follows:

$$y = c_1\, e^{2x} + c_2\, e^{3x} + \frac{1}{6}t^2 + \frac{5}{18}\ t + \frac{19}{108}$$

The general solution is solved by using the initial value of $y(0) = 1$, and $y'(0) = -2$ to find the value of c_1 and c_2 so that the exact solution is obtained as follows:

$$y = \frac{513}{108} e^{2x} - \frac{212}{54} e^{3x} + \frac{1}{6} t^2 + \frac{5}{18} t + \frac{19}{108}$$

3.2 Calculation Results of Exact Solution and Numerical Solution

In this section, the comparison of exact solution and numerical solution for case 1 and case 2 for non-homogeneous second order differential equations calculated using Euler, RK4, ABM4, and RKCoM4 methods with the help of Matlab application is studied. The calculation is done for several Step values, including: h=0.1, h=0.025, and h=0.00001.

a. Calculation Results for Case 1 Non Homogeneous 2nd order PD

From the results of calculations and numerical simulations of equation 1 of non-homogeneous second-order PD with homogeneous parts in the form of trigonometric and exponential functions, the calculation results of the exact solution and numerical solution are obtained respectively with calculations at 3 different Step (h) values, including when the value of h=0.1, h=0.025, and h=0.00001, the results are obtained in **Table 1-Table 3** and the calculation results of the error value in **Table 4-Table 6**.

Table 1. Comparison of exact and numerical solutions for equation 1 using Euler, RK4, ABM4, RKCoM4 with h=0.1.

i	*xi*	*y_Eksak*	*y_Euler*	*y_RK4*	*y_ABM4*	*y_RK4CoM*
1	0.0	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
2	0.1	-0.7736	-0.9000	-0.7740	-0.7740	-0.7675
3	0.2	-0.1566	-0.6000	-0.1575	-0.1575	-0.1447
4	0.3	1.1564	0.0382	1.1541	1.1541	1.1753
5	0.4	3.6604	1.2260	3.6559	3.6552	3.6883
6	0.5	8.1489	3.2824	8.1401	8.1382	8.1885
7	0.6	15.8783	6.6847	15.8621	15.8580	15.9332
8	0.7	28.8226	12.1402	28.7939	28.7857	28.8977
9	0.8	50.0609	20.6908	50.0109	49.9959	50.1623
10	0.9	84.3677	33.8620	84.2823	84.2558	84.5038
11	1.0	139.1114	53.8772	138.9676	138.9222	139.2930

Table 2. Comparison of exact and numerical solutions for equation 1 using Euler, RK4, ABM4, RKCoM4 with h=0.025.

i	*xi*	*y_Eksak*	*y_Euler*	*y_RK4*	*y_ABM4*	*y_RK4CoM*
1	0.00	-1	-1	-1	-1	-1
5	0.10	-0.7736	-0.8161	-0.7736	-0.7736	-0.7732
9	0.20	-0.1566	-0.3072	-0.1566	-0.1566	-0.1557
13	0.30	1.1564	0.7713	1.1564	1.1564	1.1576
17	0.40	3.6604	2.8101	3.6604	3.6605	3.6621
21	0.50	8.1489	6.4236	8.1488	8.1489	8.1509
25	0.60	15.8783	12.5692	15.8782	15.8784	15.8806
29	0.70	28.8226	22.7251	28.8225	28.8228	28.8249

i	xi	y_Eksak	y_Euler	y_RK4	y_ABM4	y_RK4CoM
33	0.80	50.0609	39.1585	50.0606	50.0612	50.0625
37	0.90	84.3677	65.3262	84.3672	84.3683	84.3675
41	1.00	139.1114	106.4736	139.1106	139.1124	139.1074

Table 3. Comparison of exact and numerical solutions for equation 1 using Euler, RK4, ABM4, RKCoM4 with h=0.00001.

i	*Xi*	*y_Eksak*	*y_Euler*	*y_RK4*	*y_ABM4*	*y_RK4CoM*
1	0.00	-1	-1	-1	-1	-1
10001	0.10	-0.7736	-0.7737	-0.7736	-0.7736	-0.7736
20001	0.20	-0.1566	-0.1566	-0.1566	-0.1566	-0.1566
30001	0.30	1.1564	1.1562	1.1564	1.1564	1.1564
40001	0.40	3.6604	3.6601	3.6604	3.6604	3.6604
50001	0.50	8.1489	8.1481	8.1489	8.1489	8.1489
60001	0.60	15.8783	15.8768	15.8783	15.8783	15.8783
70001	0.70	28.8226	28.8198	28.8226	28.8226	28.8226
80001	0.80	50.0609	50.0557	50.0609	50.0609	50.0609
90001	0.90	84.3677	84.3586	84.3677	84.3677	84.3677
100001	1.00	139.1114	139.0958	139.1114	139.1114	139.1114

From the calculation of the exact solution and numerical solution (Table 1-Table 3) using the Euler method, the 4th order Runge-Kutta method, the Adam-Bashforth-Moulton (ABM4) method and the 4th order Runge-Kutta Contra Harmonic Mean (RK4CoM) method, it is obtained that when the value of h=0.1 with the 11th iteration, the RK4 solution value (139.1106) has a value closer to the exact value (139.1114), the same condition occurs when the value of h=0.025. While during the calculation for the value of h=0.00001 at the 10001th iteration, the values of RK4, ABM4 and RK4CoM have a value (139.1114) which is very similar to the exact value (139.1114) compared to the Euler method.

Table 4. Comparison of error values for equation 1 with Euler, RK4, ABM4, and RKCoM4 methods with h=0.1.

I	*Xi*	*Error_Euler*	*Error_RK4*	*Error_ABM4*	*Error_RKCoM4*
1	0.00	0.000000	0.000000	0.000000	0.000000
2	0.10	0.126352	0.000308	0.000308	0.006101
3	0.20	0.443438	0.000959	0.000959	0.011904
4	0.30	1.118166	0.002228	0.002228	0.018910
5	0.40	2.434496	0.004580	0.005211	0.027886
6	0.50	4.866457	0.008792	0.010696	0.039596
7	0.60	9.193600	0.016144	0.020334	0.054943
8	0.70	16.68244	0.028738	0.036928	0.075073
9	0.80	29.37008	0.049984	0.065004	0.101471
10	0.90	50.50561	0.085389	0.111857	0.136107
11	1.00	85.23416	0.143795	0.189184	0.181657

Table 5. **Comparison of error values for equation 1 with Euler, RK4, ABM4, and RKCoM4 methods with the value of h=0.025**

I	*Xi*	*Error_Euler*	*Error_RK4*	*Error_ABM4*	*Error_RKCoM4*
1	0.00	0.000000	0.000000	0.000000	0.000000
5	0.10	0.042439	0.000002	0.000001	0.000453
9	0.20	0.150674	0.000005	0.000003	0.000836
13	0.30	0.385034	0.000011	0.000010	0.001232
17	0.40	0.850361	0.000023	0.000025	0.001644
21	0.50	1.725239	0.000045	0.000054	0.002031
25	0.60	3.309093	0.000082	0.000104	0.002300
29	0.70	6.097557	0.000146	0.000193	0.002275
33	0.80	10.90241	0.000253	0.000345	0.001642
37	0.90	19.04144	0.000432	0.000604	0.000138
41	1.00	32.63772	0.000727	0.001035	0.003958

Table 6. **Table 6. Comparison of error values for equation 1 with Euler, RK4, ABM4, and RKCoM4 methods with the value of h=0.00001**

I	*Xi*	*Error_Euler*	*Error_RK4*	*Error_ABM4*	*Error_RKCoM4*
1	0.0	0	0	0	0
10001	0.1	0.000019	0	0	8.1E-11
20001	0.2	0.000067	0	0	1.48E-10
30001	0.3	0.000174	0	1E-14	2.15E-10
40001	0.4	0.000389	3E-14	3E-14	2.82E-10
50001	0.5	0.000794	8E-14	9E-14	3.37E-10
60001	0.6	0.001535	1.4E-13	1.5E-13	3.6E-10
70001	0.7	0.002849	2.7E-13	3.1E-13	3.12E-10
80001	0.8	0.005131	5.5E-13	6.3E-13	1.26E-10
90001	0.9	0.009029	1.35E-12	1.46E-12	3.15E-10
100001	1.0	0.015595	2.5E-12	2.67E-12	1.19E-09

From the calculation of the error value of numerical solutions using the Euler method, the 4th order Runge-Kutta method, the Adam-Bashforth-Moulton (ABM4) method and the 4th order Runge-Kutta Contra Harmonic Mean (RK4CoM) method (**Table 4** - **Table 6**), it is obtained that when the value of h=0.1 with the 11th iteration, the smallest error value is obtained in the RK4 method (0.143795) followed by the ABM4 method (0.189184) and RK4CoM (0.181657). Meanwhile, the Euler method (85.2342) has a rather high Error value than the other three methods. Similar conditions also occur in the calculation of errors with values of h=0.025 and h=0.00001, where the Euler error value is greater than the error values in RK4, ABM, and RK4CoM. Especially for the value of h=0.00001 at the 10001th iteration, the error value of the RK4 and ABM4 methods is smaller and has a similar value of 3E-12.

b. Calculation Results for Case 2 Non Homogeneous 2nd order PD

In this part of case 2, a non-homogeneous second-order PD with a homogeneous part in the form of a quadratic function is used.

Tabel 7. **Comparison of exact and numerical solutions for Equation 2 using Euler, RK4, ABM4, RKCoM4 with h=0.1.**

I	*Xi*	*y_Eksak*	*y_Euler*	*y_RK4*	*y_ABM4*	*y_RKCoM4*
1	0.0	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.1	0.7076	0.8000	0.7077	0.7077	0.7057
3	0.2	0.1708	0.4400	0.1710	0.1710	0.1670
4	0.3	-0.7269	-0.1479	-0.7265	-0.7265	-0.7329
5	0.4	-2.1495	-1.0557	-2.1488	-2.1488	-2.1579
6	0.5	-4.3265	-2.4078	-4.3252	-4.3254	-4.3376
7	0.6	-7.5773	-4.3708	-7.5753	-7.5756	-7.5916
8	0.7	-12.3455	-7.1684	-12.3423	-12.343	-12.3635
9	0.8	-19.2445	-11.099	-19.2394	-19.2406	-19.2665
10	0.9	-29.1200	-16.5599	-29.1122	-29.1142	-29.1466
11	1.0	-43.1359	-24.0788	-43.1242	-43.1274	-43.1676

From the results of numerical calculations and simulations, the calculation results of the exact solution and the numerical solution are obtained respectively with calculations at 3 different Step (h) values, including when the value of h = 0.1, h = 0.025, and h = 0.00001, the numerical solution results are as shown in Table 7 - Table 9 and the calculation results of the error values in **Table 10** - **Table 12**.

Table 8. **Comparison of exact and numerical solutions for Equation 2 using Euler, RK4, ABM4, RKCoM4 with h=0.025.**

i	*xi*	*y_Eksak*	*y_Euler*	*y_RK4*	*y_ABM4*	*y_RKCoM4*
1	0.0	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.1	0.7076	0.7357	0.7076	0.7076	0.7075
9	0.2	0.1708	0.2535	0.1708	0.1708	0.1705
13	0.3	-0.7269	-0.5472	-0.7269	-0.7269	-0.7273
17	0.4	-2.1495	-1.8065	-2.1495	-2.1495	-2.1501
21	0.5	-4.3265	-3.7184	-4.3265	-4.3265	-4.3271
25	0.6	-7.5773	-6.5504	-7.5773	-7.5773	-7.5781
29	0.7	-12.3455	-10.6699	-12.3455	-12.3456	-12.3465
33	0.8	-19.2445	-16.5803	-19.2444	-19.2445	-19.2455
37	0.9	-29.1200	-24.9684	-29.1199	-29.1200	-29.1210
41	1.0	-43.1359	-36.7704	-43.1359	-43.1360	-43.1369

Tabel 9. **Comparison of exact and numerical solutions for equation 2 using Euler, RK4, ABM4, RKCoM4 with h=0.00001**

i	*xi*	*y_Eksak*	*y_Euler*	*y_RK4*	*y_ABM4*	*y_RKCoM4*
1	0.0	1.0000	1.0000	1.0000	1.0000	1.0000
10001	0.1	0.7076	0.7076	0.7076	0.7076	0.7076
20001	0.2	0.1708	0.1708	0.1708	0.1708	0.1708
30001	0.3	-0.7269	-0.7268	-0.7269	-0.7269	-0.7269
40001	0.4	-2.1495	-2.1494	-2.1495	-2.1495	-2.1495

i	*xi*	*y_Eksak*	*y_Euler*	*y_RK4*	*y_ABM4*	*y_RKCoM4*
50001	0.5	-4.3265	-4.3262	-4.3265	-4.3265	-4.3265
60001	0.6	-7.5773	-7.5769	-7.5773	-7.5773	-7.5773
70001	0.7	-12.3455	-12.3448	-12.3455	-12.3455	-12.3455
80001	0.8	-19.2445	-19.2433	-19.2445	-19.2445	-19.2445
90001	0.9	-29.1200	-29.1181	-29.1200	-29.1200	-29.1200
100001	1.0	-43.1359	-43.1331	-43.1359	-43.1359	-43.1359

From the calculation of the comparison of exact solutions and numerical solutions (**Table 4-Table 6**) using the Euler method, RK 4 method, ABM4 method and RK4CoM method, it is obtained that when the value of h=0.1 with the 11th iteration, the RK4 solution value (-43.1242), ABM4 (-43.1274) and RK4CoM (-43.1676) have values closer to the exact value (-43.1359). When the value of h=0.025, the RK4 error value (-43.1359) is closer to the exact value (-43.1359) than the ABM4 (-43.1360) and RK4CoM (-43.1369) methods. While during the calculation for the value of h = 0.00001 at the 10001th iteration, the RK4, ABM4 and RK4CoM values have a value of -43.1359) which is very similar to the exact value (-43.1359) compared to the Euler method (-43.1331).

Table 10. Comparison of error values for equation 1 with Euler, RK4, ABM4, and RKCoM4 methods with the value of h=0.1

I	xi	Error_Euler	Error_RK4	Error_ABM4	Error_RKCoM4
1	0.0	0.000000	0.000000	0.000000	0.000000
2	0.1	0.092412	0.000072	0.000072	0.001907
3	0.2	0.269188	0.000196	0.000196	0.003835
4	0.3	0.578996	0.000403	0.000403	0.005955
5	0.4	1.093760	0.000735	0.000677	0.008366
6	0.5	1.918708	0.001255	0.001084	0.011136
7	0.6	3.206512	0.002054	0.001692	0.014322
8	0.7	5.177114	0.003265	0.002586	0.017963
9	0.8	8.145447	0.005078	0.003892	0.022081
10	0.9	12.56010	0.007770	0.005789	0.026667
11	1.0	19.05711	0.011732	0.008528	0.031671

Table 11. Comparison of error values for Equation 1 with Euler, RK4, ABM4, and RKCoM4 methods with the value of h=0.025

I	xi	Error_Euler	Error_RK4	Error_ABM4	Error_RKCoM4
1	0.0	0.000000	0.000000	0.000000	0.000000
5	0.1	0.028069	0.000000	0.000000	0.000135
9	0.2	0.082640	0.000001	0.000001	0.000265
13	0.3	0.179647	0.000002	0.000002	0.000400
17	0.4	0.342974	0.000003	0.000005	0.000541
21	0.5	0.608045	0.000006	0.000009	0.000686
25	0.6	1.026927	0.000010	0.000015	0.000827
29	0.7	1.675596	0.000015	0.000025	0.000953
33	0.8	2.664197	0.000024	0.000040	0.001040

I	xi	Error_Euler	Error_RK4	Error_ABM4	Error_RKCoM4
37	0.9	4.151543	0.000037	0.000063	0.001055
41	1.0	6.365510	0.000055	0.000097	0.000947

Table 12. Comparison of error values for Equation 1 with Euler, RK4, ABM4, and RKCoM4 methods with the value of h=0.00001

I	xi	Error_Euler	Error_RK4	Error_ABM4	Error_RKCoM4
1	0.0	0	1E-16	1E-16	1E-16
10001	0.1	0.000012	2.9E-15	2.8E-15	2.33E-11
20001	0.2	0.000036	3.2E-15	3.1E-15	4.57E-11
30001	0.3	0.000078	1.41E-14	1.35E-14	6.86E-11
40001	0.4	0.000150	3.69E-14	3.64E-14	9.22E-11
50001	0.5	0.000266	5.24E-14	5.06E-14	1.16E-10
60001	0.6	0.000452	8.08E-14	7.82E-14	1.39E-10
70001	0.7	0.000741	1.63E-13	1.58E-13	1.58E-10
80001	0.8	0.001183	1.95E-13	1.88E-13	1.68E-10
90001	0.9	0.001851	4.26E-13	4.16E-13	1.64E-10
100001	1.0	0.002852	9.24E-14	7.82E-14	1.35E-10

From the calculation of the error value of the numerical solution for case 2 using the Euler method, RK4 method, ABM4 method and RK4CoM method in Table 10 - Table 12. It can be obtained that when the value of h=0.1 with the 11th iteration, the smallest error value is in the ABM4 method (0.0085283) compared to the RK4 method (0.0117320) and RK4CoM (0.0316710). Meanwhile, the Euler method (19.0571) has a rather high error value than the other three methods. For the calculation of the error value when the value of h=0.025 obtained Error_RK4 (0.0000554) is smaller than Error_ABM (0.0000968), Error_RK4CoM (0.0009466) and Erro_Euler (6.3655). Meanwhile, when calculating the error value with the value of h=0.00001, Error_ABM4 (8E-14) is smaller than Error_RK4 (9E-14), Error_RK4CoM (1.3502E-10) and Error_Euler (0.00285).

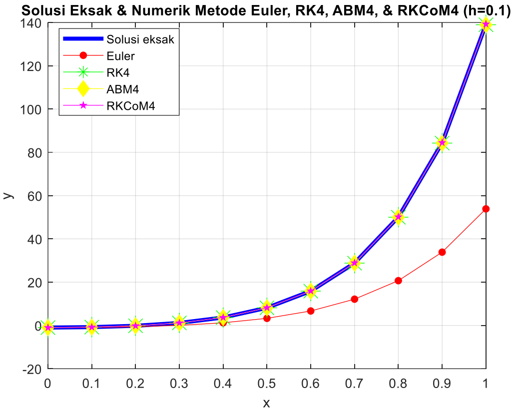
Table 13. Comparison of Mean Absolute Error (MAE) of Euler, RK4, ABM4 and RKCoM4 Methods with several different values of h with 2 different Non Homogeneous PD

Metode		MAE		
		$h = 0.1$	$h = 0.025$	$h = 0.00001$
Euler	Persamaan 1	18.1795273048	6.00557545854	0.00270916447119
RK4		0.0309923868	0.00013897400	0.00000000000038
ABM4		0.0402461859	0.00018958009	0.00000000000042
RKCoM4		0.0594225709	0.00145978711	0.00000000026564
Euler	Persamaan 2	4.73630428253	1.41765887083	0.00060938569606
RK4		0.00296005425	0.00001282598	0.00000000000011
ABM4		0.00226542598	0.00002132841	0.00000000000010
RKCoM4		0.01308202724	0.00063601742	0.00000000010475

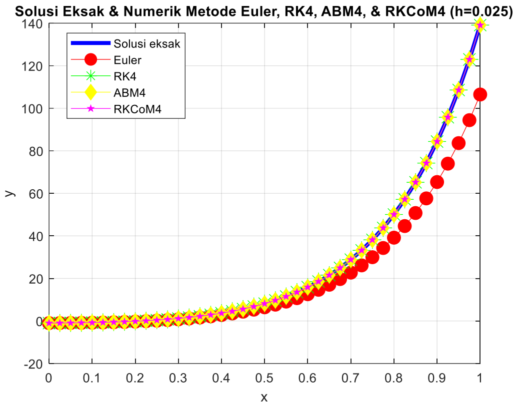
It can be obtained that the accuracy of the RK4 method and the ABM4 method has a very high accuracy with a very small MAE value for both cases of non-homogeneous second-order PD compared to the Euler and RK4CoM methods. However, for both cases of different PD forms, it is found that at the value of h=0.00001, RK4<AMB4 for the first case equation, but in the second case for the value of h=0.00001, RK4>AMB4, with a very small error difference value.

3.3 Results of Numerical Simulation of Initial Value Problem with Euler Method, RK4, ABM4, RKCoM4

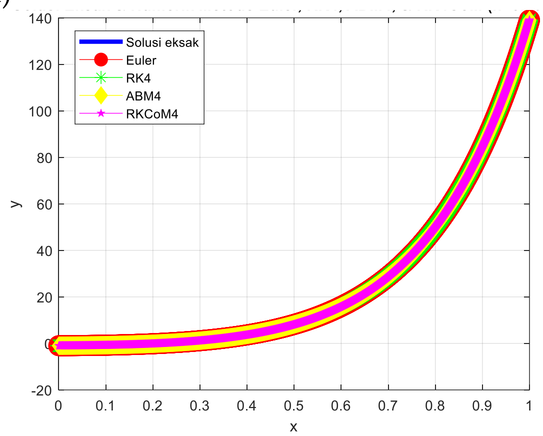
In this section, non-homogeneous second-order PD is simulated for the case of the first and second equations using four numerical methods, including: Euler, RK4, ABM4, and RK4CoM methods. So that the characteristics and graphs or trajectories of each method can be seen for each value of h=0.1, h=0.025, and h=0.00001.



(a)



(b)



(c)

Figure 1. Comparison of Exact Solution and Numerical Solution for the case of Equation 1 by Euler, RK4, ABM4, and RKCoM4 methods with different h values, (a) h=0.1, (b) h=0.025, (c) h=0.00001

In **Figure 1**, the numerical simulation results show a comparison of the exact solution and the numerical solution, where the smaller the Step value (h), the tendency of the trajectories of the four methods will look closer or closer to each other.

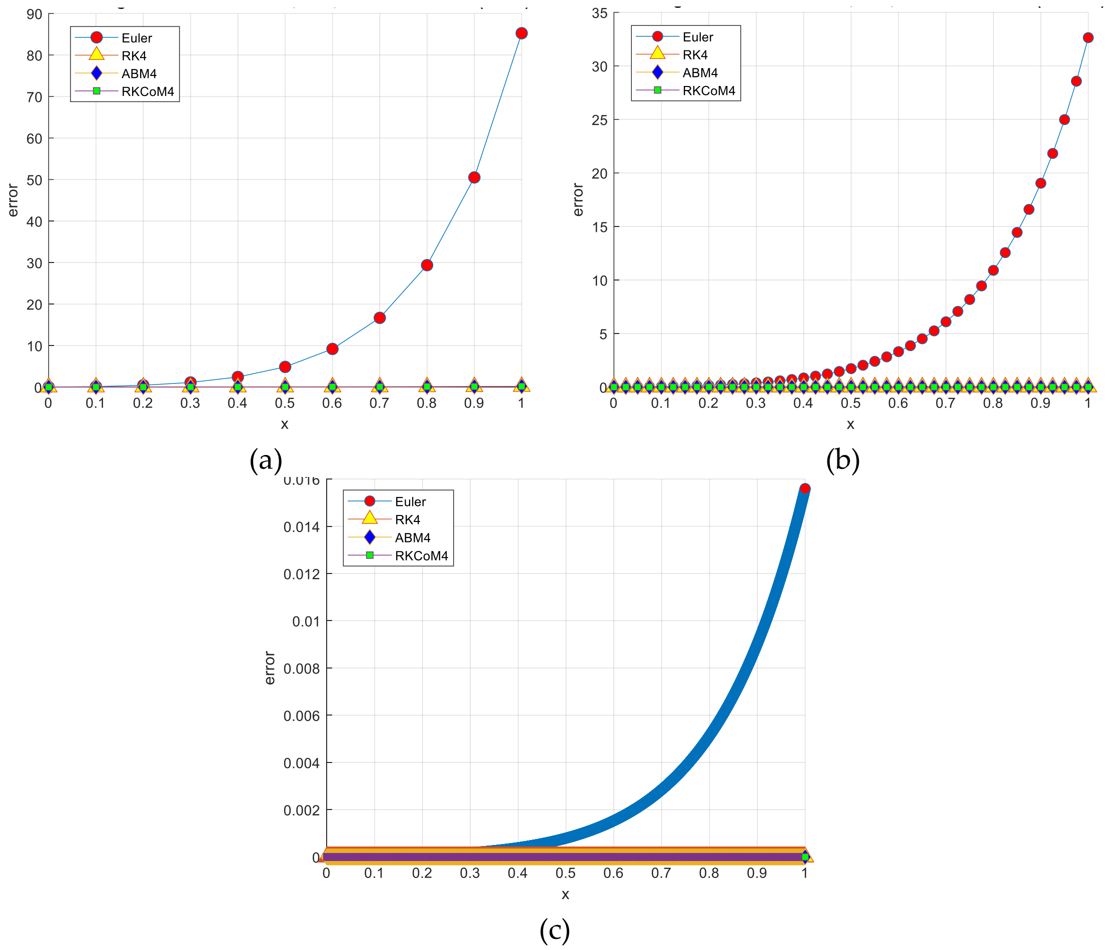
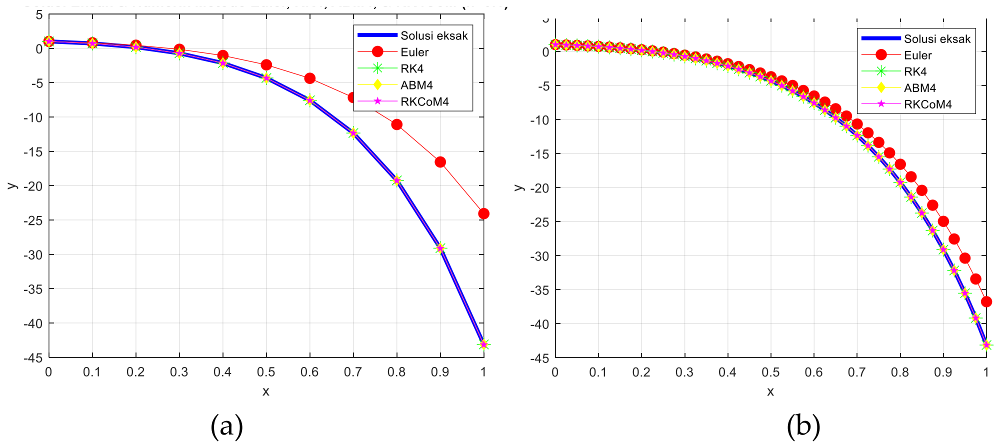
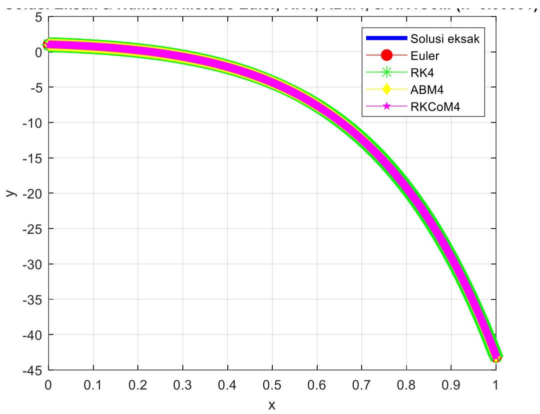


Figure 2. Comparison of errors in numerical solutions for the case of equation 1 by Euler, RK4, ABM4, and RKCoM4 methods with different h values, (a) h=0.1, (b) h=0.025, (c) h=0.00001.

In Figure 2, it can be seen that the comparison of the error values of the RK4, ABM4 and RK4CoM methods is closer than the error value of the Euler method which is further away. As the Step value increases, the density of the points is more similar to each other.

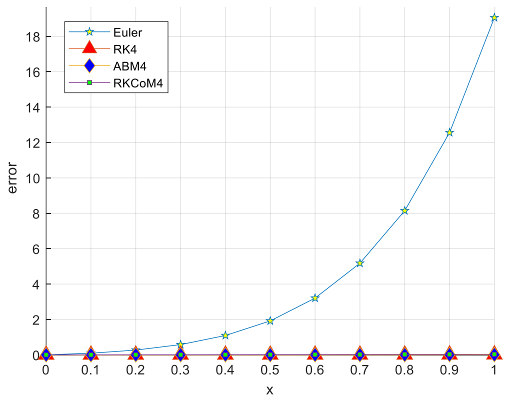




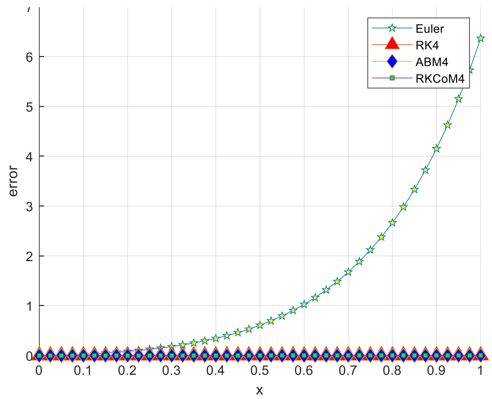
(c)

Figure 3. Comparison of Exact Solution and Numerical Solution for the case of equation 2 by Euler, RK4, ABM4, and RKCoM4 methods with different h values, (a) h=0.1, (b) h=0.025, (c) h=0.00001.

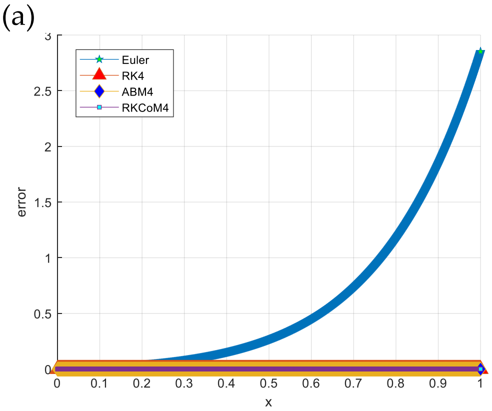
In Figure 3, the numerical simulation results show that the comparison of the exact solution and the numerical solution has the same pattern where when the value of h=0.1 is seen the three methods RK4, ABM4, and RKCoM4 on the same trajectory but for the Euler method is slightly different trajectory. When the value of h=0.00001 with very close points (small distance) is seen the four methods will be on the same trajectory.



(a)



(b)



(c)

Figure 4. Comparison of errors in numerical solutions for the case of equation 2 by Euler, RK4, ABM4, and RKCoM4 methods with different h values, (a) h=0.1, (b) h=0.025, (c) h=0.00001

In Figure 4, it can be seen that since the value of h=0.1, the comparison of the error values of the RK4, ABM4 and RK4CoM methods is much closer and even looks to be in one trajectory compared to the error value of the Euler method which is getting further away. As the Step value increases, the density of the points is more similar to each other and for Step values h=0.025. and h=0.00001, the trajectory of each method has the same pattern.

4. CONCLUSION

The conclusions that can be drawn in the results and discussion section are the results of the calculation of the Mean Absolute Error (MAE) value of the numerical methods Euler, RK4, ABM4 and RKCoM4 show that the numerical solution values of the RK4 and ABM4 methods have a much smaller MAE value or a better level of accuracy than the numerical solution values in the Euler and RKCoM4 methods at each Step value (h=0.1; h=0.025 & h=0.00001). This means that the RK4 and ABM4 methods have a high degree of similarity with the exact value. However, in some specific cases in non-homogeneous second-order PD, there are still differences in the accuracy of the numerical solution values of the RK4 and ABM4 methods, so it cannot be concluded which method is better than these two methods.

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