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THE SUFFICIENT CONDITIONS FOR A MULTIPLICATIVE DERIVATION IN THE JORDAN RING TO BE ADDITIVE

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Abstract

Derivation is a mapping from a set to itself. There are two types of derivations in rings: ordinary derivation and Jordan derivation. Given a triangular matrix ring T, a non-associative ring can be formed, known as a Jordan ring T. Subsequently, on the Jordan ring T, a derivation can be defined, referred to as derivation in the Jordan ring T. This paper provides the conditions that must be met for a multiplication derivation on the Jordan ring T to be additive. Furthermore, the ring T must be 2-torsion-free so that the derivation on the Jordan ring becomes a Jordan derivation on the ring T.

Keywords: *Derivation, Jordan Derivation, Triangular Matrix Ring.*

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1. INTRODUCTION

In the structure of abstract algebra, it is known that a group is a non-empty set in which an operation $'*'$ is defined that satisfies the axioms of closure, associativity, the existence of an identity element, and the existence of an inverse for each element [1], [2]. In addition to groups, abstract algebra also recognizes the ring structure. A ring is a non-empty set equipped with two operations, namely addition $(+)$ and multiplication $(\cdot)$. A set $\boldsymbol{R} \neq \emptyset$ is called a ring if under addition $\boldsymbol{R}$ forms an abelian group, under multiplication $\boldsymbol{R}$ forms a semigroup, and the two operations satisfy the left and right distributive properties [3]. Briefly, ring $\boldsymbol{R}$ under the operations of addition and multiplication is denoted by $(\boldsymbol{R},+,\cdot)$[4].

In every ring R, the multiplication operation must be associative. The term 'associative ring' refers to a ring with this property. The associative property in an associative ring can be omitted, resulting in an algebraic structure called a non-associative ring [5].

An associative ring R can form a new ring called a Jordan ring. The Jordan ring of R is a new ring formed by defining the multiplication in the ring R as $a \circ b = ab + ba$ for all $a, b \in R$, where is the multiplication between a and b in the associative ring R itself [6].

The concept of derivative in calculus uses polynomial rings $\mathbb{Z}[x]$ as a domain and codomain of the derivative function for each $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ with $a_i \in \mathbb{Z}$ and $i = 0,1,2,\ldots,n$ where $n \in \mathbb{N} \cup \{0\}$ holds $\frac{d}{dx}\big(f(x)\big) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in \mathbb{Z}[x]$. There is the Leibnitz's rule that for every $f(x), g(x) \in \mathbb{Z}[x]$ holds $\frac{d}{dx}\big(f(x)+g(x)\big) = \frac{d}{dx}\ \big(f(x)\big) + \frac{d}{dx}\ \big(g(x)\big) \in \mathbb{Z}[x]$ and $\frac{d}{dx}\big(f(x)\cdot g(x)\big) = \frac{d}{dx}\ \Big(f(x)\cdot g(x) + f(x)\cdot \frac{d}{dx}\ \big(g(x)\big)\Big) \in \mathbb{Z}[x]$, as described in [16]. The derivation is known in rings. A derivation is a mapping from a set to itself. In rings, there are two kinds of derivations: ordinary derivation and Jordan derivation. Given an arbitrary ring A, whether associative or not [7], an ordinary derivation is a mapping φ from the ring to itself such that for all $r, s \in A$, it satisfies $\varphi(r+s) = \varphi(r) + \varphi(s)$ and $\varphi(rs) = \varphi(r)s + r\varphi(s)$ [8],[9]. Furthermore, a Jordan derivation is an additive mapping from the associative ring to itself that satisfies $\varphi(a^2) = \varphi(a)a + a\varphi(a)$ for all $a, b \in R$ [10],[11],[12]. Every derivation is a Jordan derivation. However, a Jordan derivation is not in general a derivation [13].

A Jordan ring formed from an associative ring R can form a derivation on the Jordan ring R, namely for all $a, b \in R$, $\varphi(a \circ b) = \varphi(a) \circ b + a \circ \varphi(b)$. Furthermore, when expressed using the structure of the associative ring R, the derivation φ on the Jordan ring R is an additive mapping that satisfie $\varphi(ab + ba) = \varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a)$ for all $a, b \in R$ [10].

Furthermore, one example of an associative ring is the triangular matrix ring [14] . A triangular matrix with entries satisfying certain conditions and equipped with matrix addition and multiplication operations satisfies the ring axioms and is called a triangular matrix ring [15].

In this paper, we define the general associative ring into a triangular matrix ring. Next, an interesting topic to discuss is when a derivation on the Jordan ring is additive in the triangular matrix ring. We complete the proof of all lemmas that can be used to prove the main theorem in this research. Furthermore, conditions will be provided under which a derivation on the Jordan ring is a Jordan derivation in the triangular matrix ring. By identifying these conditions, this research will contribute to a better understanding of $n \times n$ triangular matrix ring or to find derivations in non-associative algebraic structures and may have implications for other areas of abstract algebra.

2. RESEARCH METHOD

2.1. Type of Research

The type of research used in this study is a literature review, where the researcher examines several writings related to the research, then summarizes the research results in a scientific paper.

2.2. Data Analysis

The data analysis used in this study is a study of the form of derivation and Jordan derivation and the ring of 2×2 triangular matrices which will be achieved as a reference for discussing the Jordan derivation of the ring of 2×2 triangular matrices.

3. RESULT AND DISCUSS

This section discusses the motivation behind the formation of Jordan derivation in triangular matrix ring and the form of Jordan derivation in triangular matrix ring.

3.1. Motivation for the Formation of Jordan Derivation on Triangular Matrix Ring

The triangular matrix ring T is an associative ring. Furthermore, a Jordan ring can be formed from T by defining the Jordan product operation such that $a \circ b = ab + ba, \forall a, b \in T$. Then, the derivation on the Jordan ring derived from ring T is $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(a \circ b) = \varphi(a) \circ b + a \circ \varphi(b)$ or in associative operation $\varphi(ab + ba) = \varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a),\ \forall a, b \in T$.

Based on research conducted by Martindale III (1991), which discusses a ring T that satisfies certain conditions such that every multiplicative isomorphism σ of R is additive, this can be a motivation to determine the conditions that must be fulfilled by the ring T so that the derivation multiplication on the Jordan ring T becomes additive. Furthermore, conditions will be provided so that the derivation on the Jordan ring T becomes a Jordan derivation on ring T.

3.2. Form of Jordan Derivation on Triangular Matrix Ring

Definition 1. *Given two rings $\mathfrak{R}_1$and $\mathfrak{R}_2$ ($\mathfrak{R}_1$, $\mathfrak{R}_2$ need not have an identity element) and let M be an ($\mathfrak{R}_1, \mathfrak{R}_2$)-bimodule such that*

- *i) M is a left-faithful $\mathfrak{R}_1$-module and a right-faithful $\mathfrak{R}_2$-module.*
- *ii) If $m \in M$ is such that $\mathfrak{R}_1 m \mathfrak{R}_2 = 0$then $m = 0$*

Let

$$T = \left\{ \begin{pmatrix} r_1 & m \\ 0 & r_2 \end{pmatrix} | r_1 \in \mathfrak{R}_1, r_2 \in \mathfrak{R}_2, dan\ m \in M \right\}$$

be a set of 2×2 matrix. T is a ring under the operations of addition and multiplication of matrix and is called the ring of triangular matrix.

Let ring $\boldsymbol{T} = \left\{ \begin{pmatrix} \boldsymbol{r_1} & \boldsymbol{m} \\ \boldsymbol{0} & \boldsymbol{r_2} \end{pmatrix} | \boldsymbol{r_1} \in \boldsymbol{\mathfrak{R}_1}, \boldsymbol{m} \in \boldsymbol{M}, \boldsymbol{r_2} \in \boldsymbol{\mathfrak{R}_2} \right\}$, with $\boldsymbol{\mathfrak{R}_1}$ and $\boldsymbol{\mathfrak{R}_2}$ being rings and $\boldsymbol{M}$ an $(\boldsymbol{\mathfrak{R}_1}, \boldsymbol{\mathfrak{R}_2})$-bimodule such that $\boldsymbol{M}$ is a faithful left $\boldsymbol{\mathfrak{R}_1}$-module and also a faithful right $\boldsymbol{\mathfrak{R}_2}$-module. Furthermore, if $\boldsymbol{m} \in \boldsymbol{M}$ such that $\boldsymbol{\mathfrak{R}_1 m \mathfrak{R}_2} = \boldsymbol{0}$, then $\boldsymbol{m} = \boldsymbol{0}$. Let

$$T_{11} = \left\{ \begin{pmatrix} r_1 & 0 \\ 0 & 0 \end{pmatrix} | r_1 \in \mathfrak{R}_1 \right\}, \qquad T_{12} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} | m \in M \right\}$$

and

$T_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & r_2 \end{pmatrix} | r_2 \in \mathfrak{R}_2 \right\}.$

Then it can be written as $T = T_{11} \oplus T_{12} \oplus T_{22}$. Furthermore, elements $a_{ij} \in T_{ij}$ and the corresponding elements are in $\mathfrak{R}_1, \mathfrak{R}_2$ or M. Directly, the calculation of $a_{ij} a_{kl} = 0$ if $j \neq k$ where $i, j, k \in \{1,2\}$. Next, we will explain the conditions for a derivation on the Jordan ring T to be said to be a Jordan derivation on the ring T. However, before that, we will give the definition of a k-torsion free ring as follows.

Definition 2. *A ring R is called a k-torsion free ring if $kx = 0$ leads to $x = 0$, for all $x \in \mathfrak{R}$, where $k \in \mathbb{Z}, k > 0$.*

The following theorem describes the conditions that a triangular matrix ring must satisfy in order for the multiplicative mapping of derivation on a Jordan ring T to be additive. Furthermore, we will give the condition that a derivation on the Jordan ring T is called a Jordan derivation on the ring T.

Theorem 1. *Let T is a triangular matrix ring that satisfies the following conditions:*

a) If $r_1 \mathfrak{R}_1 + \mathfrak{R}_1 r_1 = 0$ then $r_1 = 0$,

b) If $r_2 \mathfrak{R}_2 + \mathfrak{R}_2 r_2 = 0$ then $r_2 = 0$,

If the mapping $\varphi : T \to T$ satisfies $\varphi(ab + ba) = \varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a)$ for all $a, b \in T$, then the mapping φ is additive. Furthermore, if T is 2-torsion free, then φ is a Jordan derivative.

To prove Theorem 1, the following lemmas are needed by always assuming conditions a) and b) and the mapping $\varphi : T \to T$ satisfies $\varphi(ab + ba) = \varphi(a)b + a\varphi(b) + \varphi(b)a + b\varphi(a)$ for all $a, b \in T$. It is clear that $\varphi(0) = 0$.

The first lemma needed is the lemma that shows that the result of the operation of the sum of two elements, namely the element in T_{11} and the element in T_{12} is the same as the sum of the results of the operation on each separate element. In addition, the same thing will also be shown for the element in T_{22} and the element in T_{12}.

Lemma 1. *For all $s_1 \in T_{11}, r_2 \in T_{22}, p \in T_{12}$ holds*

(1) $\varphi(s_1 + m) = \varphi(s_1) + \varphi(p)$

(2) $\varphi(s_2 + m) = \varphi(s_2) + \varphi(p)$

proof.

(1) Take any $s_1 \in T_{11}, p \in T_{12}$, and $a_2 \in T_{22}$

It will be shown that $\varphi(s_1 + m) = \varphi(s_1) + \varphi(p)$

Furthermore,

Based on the mapping assumption $\varphi : T \to T$, we obtain

$\varphi[(s_1 + p)a_2 + a_2(s_1 + p)]$

$= \varphi(s_1 + p)a_2 + (s_1 + p)\varphi(a_2) + \varphi(a_2)(s_1 + p) + a_2\varphi(s_1 + p)$

On the other hand,

$$\begin{aligned} \varphi[(s_1 + p)a_2 + a_2(s_1 + p)] &= \varphi(pa_2) = \varphi(0) + \varphi(pa_2 + 0) \\ &= \varphi(s_1 a_2 + a_2 s_1) + \varphi(pa_2 + a_2 p) \\ &= \varphi(s_1)a_2 + s_1\varphi(a_2) + \varphi(a_2)s_1 + a_2\varphi(s_1) + \varphi(p)a_2 + \\ &\quad p\varphi(a_2) + \varphi(a_2)p + a_2\varphi(p) \end{aligned}$$

From these two equations it is obtained

$$\Leftrightarrow \varphi(s_1+p)a_2 + (s_1+p)\varphi(a_2) + \varphi(a_2)(s_1+p) + a_2\varphi(s_1+p)$$
$$= \varphi(s_1)a_2 + s_1\varphi(a_2) + \varphi(a_2)s_1 + a_2\varphi(s_1) + \varphi(p)a_2 + p\varphi(a_2) + \varphi(a_2)p + a_2\varphi(p)$$

$$\Leftrightarrow \varphi(s_1+p)a_2 + (s_1+p)\varphi(a_2) + \varphi(a_2)(s_1+p) + a_2\varphi(s_1+p) - \varphi(s_1)a_2 - s_1\varphi(a_2) - \varphi(a_2)s_1 - a_2\varphi(s_1) - \varphi(p)a_2 - p\varphi(a_2) - \varphi(a_2)p - a_2\varphi(p) = 0$$

$$\Leftrightarrow \varphi(s_1+p)a_2 + s_1\varphi(a_2) + p\varphi(a_2) + \varphi(a_2)s_1 + \varphi(a_2)p + a_2\varphi(s_1+p) - \varphi(s_1)a_2 - s_1\varphi(a_2) - \varphi(a_2)s_1 - a_2\varphi(s_1) - \varphi(p)a_2 - p\varphi(a_2) - \varphi(a_2)p - a_2\varphi(p) = 0$$

$$\Leftrightarrow \varphi(s_1+p)a_2 + a_2\varphi(s_1+p) - \varphi(s_1)a_2 - a_2\varphi(s_1) - \varphi(p)a_2 - a_2\varphi(p) = 0$$

$$\Leftrightarrow [\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]a_2 + a_2[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)] = 0$$

Thus

$$[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{12}a_2 = 0$$

and

$$[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{22}a_2 + a_2[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{22} = 0$$

Based on conditions ii) and b) of Definition 1 and Theorem 1, we obtain

$$[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{12} = 0$$
$$[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{22} = 0$$

Next, to complete the proof, we will show that $[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{11} = 0$.

Take any $q \in T_{12}$

Under the assumption of mapping $\varphi : T \to T$, we have

$$\varphi[(s_1+p)q + q(r_1+p)] = \varphi(s_1+p)q + (s_1+p)\varphi(q) + \varphi(q)(s_1+p) + q\varphi(s_1+p)$$

In addition,

$$\varphi[(s_1+p)q + q(r_1+p)] = \varphi(s_1q) = \varphi(s_1q+0) + \varphi(0)$$
$$= \varphi(s_1q + qs_1) + \varphi(pq + qp)$$
$$= \varphi(s_1)q + s_1\varphi(q) + \varphi(q)s_1 + q\varphi(s_1) + \varphi(p)q + p\varphi(q) + \varphi(q)p + q\varphi(p)$$

From these two equations, it is obtained

$$\Leftrightarrow \varphi(s_1+p)q + (s_1+p)\varphi(q) + \varphi(q)(s_1+p) + q\varphi(s_1+p)$$
$$= \varphi(s_1)q + s_1\varphi(q) + \varphi(q)s_1 + q\varphi(s_1) + \varphi(p)q + p\varphi(q) + \varphi(q)p + q\varphi(p)$$

$$\Leftrightarrow \varphi(s_1+p)q + (s_1+p)\varphi(q) + \varphi(q)(s_1+p) + q\varphi(s_1+p) - \varphi(s_1)q - s_1\varphi(q) - \varphi(q)s_1 - q\varphi(s_1) - \varphi(p)q - p\varphi(q) - \varphi(q)p - q\varphi(p) = 0$$

$$\Leftrightarrow \varphi(s_1+p)q + s_1\varphi(q) + p\varphi(q) + \varphi(q)s_1 + \varphi(q)p + q\varphi(s_1+p) - \varphi(s_1)q - s_1\varphi(q) - \varphi(q)s_1 - q\varphi(s_1) - \varphi(p)q - p\varphi(q) - \varphi(q)p - q\varphi(p) = 0$$

$$\Leftrightarrow \varphi(s_1+p)q + q\varphi(s_1+p) - \varphi(s_1)q - q\varphi(s_1) - \varphi(p)q - q\varphi(p) = 0$$

$$\Leftrightarrow [\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]q + q[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)] = 0$$

Therefore,

$$[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{11}p = 0.$$

Based on condition i) in Definition 1, we obtain

$$[\varphi(s_1+p) - \varphi(s_1) - \varphi(p)]_{11} = 0.$$

Thus, it is proven that $\varphi(s_1+p) = \varphi(s_1) + \varphi(p)$.

(2) Take any $a_1 \in T_{11}, s_2 \in T_{22}, p \in T_{12}$

It will be shown that $\varphi(s_2+p) = \varphi(s_2) + \varphi(p)$

Furthermore,

Based on the assumption $\varphi : T \to T$, we obtain

$\varphi[(s_2+p)a_1+a_1(s_2+p)]$
$= \varphi(s_2+p)a_1+(s_2+p)\varphi(a_1)+\varphi(a_1)(s_2+p)+a_1\varphi(s_2+p)$

On the other hand,

$$\varphi[(s_2+p)a_1+a_1(s_2+p)] = \varphi(a_1p) = \varphi(0)+\varphi(0+a_1p)$$
$$= \varphi(s_2a_1+a_1s_2)+\varphi(pa_1+a_1p)$$

$$= \varphi(s_2)a_1+s_2\varphi(a_1)+\varphi(a_1)s_2+\varphi(s_2)+\varphi(p)a_1+p\varphi(a_1)+\varphi(a_1)p+a_1\varphi(p)$$

From these two equations, it is obtained

$$\Leftrightarrow \varphi(s_2+p)a_1+(s_2+p)\varphi(a_1)+\varphi(a_1)(s_2+p)+a_1\varphi(s_2+p) = \varphi(s_2)a_1+s_2\varphi(a_1)+\varphi(a_1)s_2+a_1\varphi(s_2)+\varphi(p)a_1+p\varphi(a_1)+\varphi(a_1)p+a_1\varphi(p)$$

$$\Leftrightarrow \varphi(s_2+p)a_1+(s_2+p)\varphi(a_1)+\varphi(a_1)(s_2+p)+a_1\varphi(s_2+p)-\varphi(s_2)a_1-s_2\varphi(a_1) - \varphi(a_1)s_2-a_1\varphi(s_2)-\varphi(p)a_1-p\varphi(a_1)-\varphi(a_1)p-a_1\varphi(p)=0$$

$$\Leftrightarrow \varphi(s_2+p)a_1+s_2\varphi(a_1)+p\varphi(a_1)+\varphi(a_1)s_2+\varphi(a_1)p+a_1\varphi(s_2+p)-\varphi(s_2)a_1 - s_2\varphi(a_1)-\varphi(a_1)s_2-a_1\varphi(s_2)-\varphi(p)a_1-p\varphi(a_1)-\varphi(a_1)p - a_1\varphi(p)=0$$

$$\Leftrightarrow \varphi(s_2+p)a_1+a_1\varphi(s_2+p)-\varphi(s_2)a_1-a_1\varphi(s_2)-\varphi(p)a_1-a_1\varphi(p)=0$$

$$\Leftrightarrow [\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]a_1+a_1[\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]=0$$

Therefore

$$a_1[\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]_{12}=0$$

and

$$[\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]_{11}a_1+a_1[\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]_{11}=0.$$

Based on conditions ii) and a) of Definition 1 and Theorem 1, we obtain

$$[\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]_{12}=0$$
$$[\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]_{11}=0$$

Next, to complete the proof, we will show that $[\varphi(s_2+p)-\varphi(s_2)-\varphi(p)]_{22}=0$.

Take any $q\in T_{12}$

Under the assumption of the mapping $\varphi: T\to T$, we obtain

$$\varphi[(s_2+p)q+q(s_2+p)] = \varphi(s_2+p)q+(s_2+p)\varphi(q)+\varphi(q)(s_2+p)+q\varphi(s_2+p)$$

In addition,

$$\varphi[(s_2+p)q+q(s_2+p)] = \varphi(qs_2) = \varphi(0+qs_2)+\varphi(0)$$
$$= \varphi(s_2q+qs_2)+\varphi(pq+qp)$$

$$= \varphi(s_2)q+s_2\varphi(q)+\varphi(q)s_2+q\varphi(s_2)+\varphi(p)q+p\varphi(q)+\varphi(q)p+q\varphi(p)$$

From these two equations, it is obtained

$$\Leftrightarrow \varphi(s_2+p)q+(s_2+p)\varphi(q)+\varphi(q)(s_2+p)+q\varphi(s_2+p) = \varphi(s_2)q+s_2\varphi(q)+\varphi(q)s_2+q\varphi(s_2)+\varphi(p)q+p\varphi(q)+\varphi(q)p+q\varphi(p)$$

$$\Leftrightarrow \varphi(s_2+p)q+(s_2+p)\varphi(q)+\varphi(q)(s_2+p)+q\varphi(s_2+p)-\varphi(s_2)q-s_2\varphi(q) - \varphi(q)s_2-q\varphi(s_2)-\varphi(p)q-p\varphi(q)-\varphi(q)p-q\varphi(p)=0$$

$$\Leftrightarrow \varphi(s_2+p)q+s_2\varphi(q)+p\varphi(q)+\varphi(q)s_2+\varphi(q)p+q\varphi(s_2+p)-\varphi(s_2)q-s_2\varphi(q) - \varphi(q)s_2-q\varphi(s_2)-\varphi(p)q-p\varphi(q)-\varphi(q)p-q\varphi(p)=0$$

$\Leftrightarrow \varphi(s_2+p)q + q\varphi(s_2+p) - \varphi(s_2)q - q\varphi(s_2) - \varphi(p)q - q\varphi(p) = 0$
$\Leftrightarrow [\varphi(s_2+p) - \varphi(s_2) - \varphi(p)]q + q[\varphi(s_2+p) - \varphi(s_2) - \varphi(p)] = 0$
Therefore,
$q[\varphi(s_2+p) - \varphi(s_2) - \varphi(p)]_{22} = 0.$
Based on condition i) of Definition 1, we have
$[\varphi(s_2+p) - \varphi(s_2) - \varphi(p)]_{22} = 0.$
Thus, it is proven that $\varphi(s_2+p) = \varphi(s_2) + \varphi(p)$.

■

Lemma 2. *For all $s_1 \in T_{11}, s_2 \in T_{22}$, and $p, q \in T_{12}$, it holds that*
$\varphi(s_1p + qs_2) = \varphi(s_1p) + \varphi(qr_2)$.

Proof.

Take any $s_1 \in T_{11}, r_2 \in T_{22}$, and $p, q \in T_{12}$
It will be shown that $\varphi(s_1p + qs_2) = \varphi(s_1p) + \varphi(qs_2)$.
Furthermore,
$s_1p + qs_2$ can be expressed as $(s_1+q)(p+s_2) + (p+s_2)(s_1+q)$.
So that
$\varphi(s_1p + qs_2) = \varphi\big((s_1+q)(p+s_2) + (p+s_2)(s_1+q)\big)$

$= \varphi(s_1+q)(p+s_2) + (s_1+q)\varphi(p+s_2) + \varphi(p+s_2)(s_1+q) +$
$(p+s_2)\varphi(s_1+q)$

Based on Lemma 1, we get

$= \big(\varphi(s_1) + \varphi(q)\big)(p+s_2) + (s_1+q)\big(\varphi(p) + \varphi(s_2)\big) + \big(\varphi(p) + \varphi(s_2)\big)(s_1 +$
$q) + (p+s_2)\big(\varphi(s_1) + \varphi(q)\big)$

$$\begin{aligned}
&= \varphi(s_1)(p+s_2) + \varphi(q)(p+s_2) + (s_1+q)\varphi(p) + (s_1+q)\varphi(s_2) \\
&\qquad + \varphi(p)(s_1+q) + \varphi(s_2)(s_1+q) + (p+s_2)\varphi(s_1) \\
&\qquad + (p+s_2)\varphi(q) \\
&= \varphi(s_1)p + \varphi(s_1)s_2 + \varphi(q)p + \varphi(q)s_2 + s_1\varphi(p) + q\varphi(p) + s_1\varphi(s_2) + q\varphi(s_2) \\
&\qquad + \varphi(p)s_1 + \varphi(p)q + \varphi(s_2)s_1 + \varphi(s_2)q + p\varphi(s_1) + s_2\varphi(s_1) \\
&\qquad + p\varphi(q) + s_2\varphi(q)
\end{aligned}$$
$= [\varphi(s_1)p + s_1\varphi(p) + \varphi(p)s_1 + p\varphi(s_1)] + [\varphi(s_1)s_2 + s_1\varphi(s_2) + \varphi(s_2)s_1 +$
$s_2\varphi(s_1)] + [\varphi(q)p + q\varphi(p) + \varphi(p)q + p\varphi(q)] + [\varphi(q)s_2 + q\varphi(s_2) +$
$\varphi(s_2)q + s_2\varphi(q)]$

$$\begin{aligned}
&= \varphi(s_1p + ps_1) + \varphi(s_1s_2 + s_2s_1) + \varphi(qp + pq) + \varphi(qs_2 + s_2q) \\
&= \varphi(s_1p + ps_1) + \varphi(qs_2 + s_2q) \\
&= \varphi(s_1p) + \varphi(qs_2).
\end{aligned}$$
So, it is proven that $\varphi(s_1p + qs_2) = \varphi(s_1p) + \varphi(qs_2)$.

■

Lemma 3. *For all $s_2 \in T_{22}$ and $p, q \in T_{12}$, it holds that $\varphi(ps_2 + qs_2) = \varphi(ps_2) + \varphi(qs_2)$.*

Proof.

Take any $s_1 \in T_{22}, s_2 \in T_{22}$, and $p, q \in T_{12}$
It will be shown that $\varphi(ps_2 + qs_2) = \varphi(ps_2) + \varphi(qs_2)$`

Furthermore,

$$\varphi[s_1((p+q)s_2)+((p+q)s_2)s_1]$$
$$=\varphi(s_1)((p+q)s_2)+s_1\varphi((p+q)s_2)+\varphi((p+q)s_2)s_1+((p+q)s_2)\varphi(s_1)$$

On the other hand, by Lemma 2, we have

$$\varphi[s_1((p+q)s_2)+((p+q)s_2)s_1]=\varphi(s_1(ps_2)+(s_1q)s_2)$$
$$=\varphi(s_1ps_2)+\varphi(s_1qs_2)``$$
$$=\varphi(s_1(ps_2)+(ps_2)s_1)+\varphi(s_1(qs_2)+(qs_2)s_1)$$
$$=\varphi(s_1)(ps_2)+s_1\varphi(ps_2)+\varphi(ps_2)s_1+(ps_2)\varphi(s_1)+\varphi(s_1)(qs_2)+s_1\varphi(qs_2)+\varphi(qs_2)s_1+(qs_2)\varphi(s_1).$$

From these two equations it is obtained

$$\Leftrightarrow \varphi(s_1)((p+q)s_2)+s_1\varphi((p+q)s_2)+\varphi((p+q)s_2)s_1+((p+q)s_2)\varphi(s_1)$$
$$=\varphi(s_1)(ps_2)+s_1\varphi(ps_2)+\varphi(ps_2)s_1+(ps_2)\varphi(s_1)+\varphi(s_1)(qs_2)+s_1\varphi(qs_2)+\varphi(qs_2)s_1+(qs_2)\varphi(s_1)$$
$$\Leftrightarrow \varphi(s_1)((p+q)s_2)+s_1\varphi((p+q)s_2)+\varphi((p+q)s_2)s_1+((p+q)s_2)\varphi(s_1)-\varphi(s_1)(ps_2)-s_1\varphi(ps_2)-\varphi(ps_2)s_1-(ps_2)\varphi(s_1)-\varphi(s_1)(qs_2)-s_1\varphi(qs_2)-\varphi(qs_2)s_1-(qs_2)\varphi(s_1)=0$$
$$\Leftrightarrow \varphi(s_1)(ps_2)+\varphi(s_1)(qs_2)+s_1\varphi((p+q)s_2)+\varphi((p+q)s_2)s_1+((p+q)s_2)+(ps_2)\varphi(s_1)+(qs_2)\varphi(s_1)-\varphi(s_1)(ps_2)-s_1\varphi(ps_2)-\varphi(ps_2)s_1-(ps_2)\varphi(s_1)-\varphi(s_1)(qs_2)-s_1\varphi(qs_2)-\varphi(qs_2)s_1-(qs_2)\varphi(s_1)=0$$
$$\Leftrightarrow s_1\varphi((p+q)s_2)+\varphi((p+q)s_2)s_1-s_1\varphi(ps_2)-\varphi(ps_2)s_1-s_1\varphi(qs_2)-\varphi(qs_2)s_1=0$$

So that

$$[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]s_1+s_1[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]=0$$

for all $s_1\in T_{11}$.

Therefore

$$s_1[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]_{12}=0$$

and

$$[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]_{11}s_1+s_1[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]_{11}=0.$$

Based on conditions ii) and a) of Definition 1 and Theorem 1 respectively, we obtain

$$[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]_{12}=0;$$
$$[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]_{11}=0.$$

To complete the proof, take any $z\in T_{12}$ and show that $[\varphi((p+q)s_2)-\varphi(ps_2)-\varphi(qs_2)]_{22}=0$.

Furthermore,

$$\varphi[((p+q)s_2)z+z((p+q)s_2)]$$
$$=\varphi((p+q)s_2)z+((p+q)s_2)\varphi(z)+\varphi(z)((p+q)s_2)+z\varphi((p+q)s_2)$$

On the other hand,

$$\varphi[((p+q)s_2)z+z((p+q)s_2)]=0$$
$$=\varphi((ps_2)z+z(ps_2))+\varphi((qs_2)z+\ z(qs_2))$$
$$=\varphi(ps_2)z+(ps_2)\varphi(z)+\varphi(z)(ps_2)+z\varphi(ps_2)+\varphi(qs_2)z+(qs_2)\varphi(z)+\varphi(z)(qs_2)+z\varphi(qs_2).$$

Then, from these two equations we obtain

$$\Leftrightarrow \varphi((p+q)s_2)z+((p+q)s_2)\varphi(z)+\varphi(z)((p+q)s_2)+z\varphi((p+q)s_2)$$
$$=\varphi(ps_2)z+(ps_2)\varphi(z)+\varphi(z)(ps_2)+z\varphi(ps_2)+\varphi(qs_2)z+(qs_2)\varphi(z)+\varphi(z)(qs_2)+z\varphi(qs_2)$$

$$\Leftrightarrow \varphi\big((p+q)s_2\big)z + \big((p+q)s_2\big)\varphi(z) + \varphi(z)\big((p+q)s_2\big) + z\varphi\big((p+q)s_2\big) - \varphi(ps_2)z - (ps_2)\varphi(z) - \varphi(z)(ps_2) - z\varphi(ps_2) - \varphi(qs_2)z - (qs_2)\varphi(z) - \varphi(z)(qs_2) - z\varphi(qs_2) = 0$$

$$\Leftrightarrow \varphi\big((p+q)s_2\big)z + (ps_2)\varphi(z) + (qs_2)\varphi(z) + \varphi(z)(ps_2) + \varphi(z)(qs_2) + z\varphi\big((p+q)s_2\big) - \varphi(ps_2)z - (ps_2)\varphi(z) - \varphi(z)(ps_2) - z\varphi(ps_2) - \varphi(qs_2)z - (qs_2)\varphi(z) - \varphi(z)(qs_2) - z\varphi(qs_2) = 0$$

$$\Leftrightarrow \varphi\big((p+q)s_2\big)z + z\varphi\big((p+q)s_2\big) - \varphi(ps_2)z - z\varphi(ps_2) - \varphi(qs_2)z - z\varphi(qs_2) = 0$$

$$\Leftrightarrow \big[\varphi\big((p+q)s_2\big) - \varphi(ps_2) - \varphi(qs_2)\big]z + z\big[\varphi\big((p+q)s_2\big) - \varphi(ps_2) - \varphi(qs_2)\big] = 0$$

Therefore,

$$z\big[\varphi\big((p+q)s_2\big) - \varphi(ps_2) - \varphi(qs_2)\big]_{22} = 0$$

For all $z \in T_{12}$.

Based on condition ii) of Definition 1, we obtain

$$\big[\varphi\big((p+q)s_2\big) - \varphi(ps_2) - \varphi(qs_2)\big]_{22} = 0$$

Thus, it is proven that $\varphi(ps_2 + qs_2) = \varphi(ps_2) + \varphi(qs_2)$.

■

Lemma 4. *For all $p, q \in T_{12}$, it holds that $\varphi(p+q) = \varphi(p) + \varphi(q)$.*

Proof.

Given any $p, q \in T_{12}$, and $s_2 \in T_{22}$

It will be shown that $\varphi(p+q) = \varphi(p) + \varphi(q)$

Furthermore,

$$\varphi[(p+q)s_2 + s_2(p+q)] = \varphi(p+q)(s_2) + (p+q)\varphi(s_2) + \varphi(s_2)(p+q) + (s_2)\varphi(p+q)$$

In addition, based on Lemma 3, we obtain

$$\begin{aligned}\varphi[(p+q)s_2 + s_2(p+q)] &= \varphi(ps_2 + qs_2)\\ &= \varphi(ps_2) + \varphi(qs_2)\\ &= \varphi(ps_2 + s_2p) + \varphi(qs_2 + s_2q)\\ &= \varphi(p)s_2 + p\varphi(s_2) + \varphi(s_2)p + s_2\varphi(p) + \varphi(q)s_2 + q\varphi(s_2) + \\ &\quad \varphi(s_2)q + s_2\varphi(q)\end{aligned}$$

From the two equations, we obtain

$$\Leftrightarrow \varphi(p+q)(s_2) + (p+q)\varphi(s_2) + \varphi(s_2)(p+q) + (s_2)\varphi(p+q) = \varphi(p)s_2 + p\varphi(s_2) + \varphi(s_2)p + s_2\varphi(p) + \varphi(q)s_2 + q\varphi(s_2) + \varphi(s_2)q + s_2\varphi(q)$$

$$\Leftrightarrow \varphi(p+q)(s_2) + (p+q)\varphi(s_2) + \varphi(s_2)(p+q) + (s_2)\varphi(p+q) - \varphi(p)s_2 - p\varphi(s_2) - \varphi(s_2)p - s_2\varphi(p) - \varphi(q)s_2 - q\varphi(s_2) - \varphi(s_2)q - s_2\varphi(q) = 0$$

$$\Leftrightarrow \varphi(p+q)(s_2) + p\varphi(s_2) + q\varphi(s_2) + \varphi(s_2)p + \varphi(s_2)q + (s_2)\varphi(p+q) - \varphi(p)s_2 - p\varphi(s_2) - \varphi(s_2)p - s_2\varphi(p) - \varphi(q)s_2 - q\varphi(s_2) - \varphi(s_2)q - s_2\varphi(q) = 0$$

$$\Leftrightarrow \varphi(p+q)(s_2) + (s_2)\varphi(p+q) - \varphi(p)s_2 - s_2\varphi(p) - \varphi(q)s_2 - s_2\varphi(q) = 0$$

$$\Leftrightarrow [\varphi(p+q) - \varphi(p) - \varphi(q)]s_2 + s_2[\varphi(p+q) - \varphi(p) - \varphi(q)] = 0$$

for all $s_2 \in T_{22}$.

Therefore

$$[\varphi(p+q) - \varphi(p) - \varphi(q)]_{12}s_2 = 0;$$

and

$$[\varphi(p+q) - \varphi(p) - \varphi(q)]_{22}s_2 + s_2[\varphi(p+q) - \varphi(p) - \varphi(q)]_{22} = 0.$$

By conditions ii) and b) of Definition 1 and Theorem 1 respectively, we have

$$[\varphi(p+q) - \varphi(p) - \varphi(q)]_{12} = 0;$$

$$[\varphi(p+q) - \varphi(p) - \varphi(q)]_{22} = 0.$$

Next,
Take any $z \in T_{12}$
It will be shown that $[\varphi(p+q) - \varphi(p) - \varphi(q)]_{11} = 0$
$\varphi(p+q)z + (p+q)\varphi(z) + \varphi(z)(p+q) + z\varphi(p+q) = \varphi[(p+q)z + z(p+q)] = 0$
$= \varphi(pz + zp) + \varphi(qz + zq)$
$= \varphi(p)z + p\varphi(z) + \varphi(z)p + z\varphi(p) + \varphi(q)z + q\varphi(z) + \varphi(z)q + z\varphi(q)$
We obtain

$$\begin{aligned}
&\Leftrightarrow \varphi(p+q)z + (p+q)\varphi(z) + \varphi(z)(p+q) + z\varphi(p+q) \\
&\qquad = \varphi(p)z + p\varphi(z) + \varphi(z)p + z\varphi(p) + \varphi(q)z + q\varphi(z) + \varphi(z)q + z\varphi(q) \\
&\Leftrightarrow \varphi(p+q)z + (p+q)\varphi(z) + \varphi(z)(p+q) + z\varphi(p+q) - \varphi(p)z - p\varphi(z) - \varphi(z)p - z\varphi(p) \\
&\qquad - \varphi(q)z - q\varphi(z) - \varphi(z)q - z\varphi(q) = 0 \\
&\Leftrightarrow \varphi(p+q)z + p\varphi(z) + q\varphi(z) + \varphi(z)p + \varphi(z)q + z\varphi(p+q) - \varphi(p)z - p\varphi(z) - \varphi(z)p \\
&\qquad - z\varphi(p) - \varphi(q)z - q\varphi(z) - \varphi(z)q - z\varphi(q) = 0 \\
&\Leftrightarrow \varphi(p+q)z + \varphi(z)q + z\varphi(p+q) - \varphi(p)z - z\varphi(p) - \varphi(q)z - z\varphi(q) = 0 \\
&\Leftrightarrow [\varphi(p+q) - \varphi(p) - \varphi(q)]z + z[\varphi(p+q) - \varphi(p) - \varphi(q)] = 0.
\end{aligned}$$

Therefore,
$[\varphi(p+q) - \varphi(p) - \varphi(q)]_{11} z = 0$ for all $z \in T_{12}$.
Based on condition i) of Definition 1, we have
$[\varphi(p+q) - \varphi(p) - \varphi(q)]_{11} = 0$.
Thus, it is proven that $\varphi(p+q) = \varphi(p) + \varphi(q)$.

■

Lemma 5. *For all $s_1, t_1 \in T_{11}, s_2, t_2 \in T_{22}$, it holds that*

(1) $\varphi(s_1 + t_1) = \varphi(r_1) + \varphi(t_1)$.
(2) $\varphi(s_2 + t_2) = \varphi(s_2) + \varphi(t_2)$.

Proof.

(1) Take any $s_1, t_1 \in T_{11}, a_2 \in T_{22}$
It will be shown that $\varphi(r_1 + t_1) = \varphi(r_1) + \varphi(t_1)$.
Furthermore,

$$\begin{aligned}
&\varphi[(s_1+t_1)a_2 + a_2(s_1+t_1)] \\
&\qquad = \varphi(s_1+t_1)a_2 + (s_1+t_1)\varphi(a_2) + \varphi(a_2)(s_1+t_1) + a_2\varphi(s_1+t_1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi[(s_1+t_1)a_2 + a_2(s_1+t_1)] = 0 \\
&= \varphi(s_1a_2 + a_2s_1) + \varphi(t_1a_2 + a_2t_1) \\
&= \varphi(s_1)a_2 + s_1\varphi(a_2) + \varphi(a_2)s_1 + a_2\varphi(s_1) + \varphi(t_1)a_2 + \\
&\quad t_1\varphi(a_2) + \varphi(a_2)t_1 + a_2\varphi(t_1).
\end{aligned}$$

From these two equations, we obtain

$$\begin{aligned}
&\Leftrightarrow \varphi(s_1+t_1)a_2 + (s_1+t_1)\varphi(a_2) + \varphi(a_2)(s_1+t_1) + a_2\varphi(s_1+t_1) \\
&\qquad = \varphi(s_1)a_2 + s_1\varphi(a_2) + \varphi(a_2)s_1 + a_2\varphi(s_1) + \varphi(t_1)a_2 + t_1\varphi(a_2) \\
&\qquad + \varphi(a_2)t_1 + a_2\varphi(t_1) \\
&\Leftrightarrow \varphi(s_1+t_1)a_2 + (s_1+t_1)\varphi(a_2) + \varphi(a_2)(s_1+t_1) + a_2\varphi(s_1+t_1) - \varphi(s_1)a_2 \\
&\qquad - s_1\varphi(a_2) - \varphi(a_2)s_1 - a_2\varphi(s_1) - \varphi(t_1)a_2 - t_1\varphi(a_2) - \varphi(a_2)t_1 \\
&\qquad - a_2\varphi(t_1) = 0 \\
&\Leftrightarrow \varphi(s_1+t_1)a_2 + s_1\varphi(a_2) + t_1\varphi(a_2) + \varphi(a_2)s_1 + \varphi(a_2)t_1 + a_2\varphi(s_1+t_1) - \varphi(s_1)a_2 \\
&\qquad - s_1\varphi(a_2) - \varphi(a_2)s_1 - a_2\varphi(s_1) - \varphi(t_1)a_2 - t_1\varphi(a_2) - \varphi(a_2)t_1 \\
&\qquad - a_2\varphi(t_1) = 0
\end{aligned}$$

$\Leftrightarrow \varphi(s_1+t_1)a_2 + a_2\varphi(s_1+t_1) - \varphi(s_1)a_2 - a_2\varphi(s_1) - \varphi(t_1)a_2 - a_2\varphi(t_1) = 0$
$\Leftrightarrow [\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]a_2 + a_2[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)] = 0$
Thus
$[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{12}a_2 = 0$
and
$[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{22}a_2 + a_2[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{22} = 0$
Based on conditions ii) and b) of Definition 1 and Theorem 1 respectively, we obtain
$[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{12} = 0;$
$[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{22} = 0.$
Furthermore, it will be shown that $[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{11} = 0$.
Take any $z \in T_{12}$
Based on the mapping assumption $\varphi : T \longrightarrow T$, we obtain
$\varphi[(s_1+t_1)z + z(s_1+t_1)]$
$= \varphi(s_1+t_1)z + (s_1+t_1)\varphi(z) + \varphi(z)(s_1+t_1) + z\varphi(s_1+t_1)$
On the other hand,
$\varphi[(s_1+t_1)z + z(s_1+t_1)] = \varphi(s_1z + t_1z)$
$= \varphi(s_1z) + \varphi(t_1z)$
$= \varphi(s_1z + zs_1) + \varphi(t_1z + zt_1)$
$= \varphi(s_1)z + s_1\varphi(z) + \varphi(z)s_1 + z\varphi(s_1) + \varphi(t_1)z + t_1\varphi(z) + \varphi(z)t_1 + z\varphi(t_1)$

Hence, we obtain
$\Leftrightarrow \varphi(s_1+t_1)z + (s_1+t_1)\varphi(z) + \varphi(z)(s_1+t_1) + z\varphi(s_1+t_1)$
$= \varphi(s_1)z + s_1\varphi(z) + \varphi(z)s_1 + z\varphi(s_1) + \varphi(t_1)z + t_1\varphi(z) + \varphi(z)t_1 + z\varphi(t_1)$
$\Leftrightarrow \varphi(s_1+t_1)z + (s_1+t_1)\varphi(z) + \varphi(z)(s_1+t_1) + z\varphi(s_1+t_1) - \varphi(s_1)z - s_1\varphi(z) - \varphi(z)s_1 - z\varphi(s_1) - \varphi(t_1)z - t_1\varphi(z) - \varphi(z)t_1 - z\varphi(t_1) = 0$
$\Leftrightarrow \varphi(s_1+t_1)z + s_1\varphi(z) + t_1\varphi(z) + \varphi(z)s_1 + \varphi(z)t_1 + z\varphi(s_1+t_1) - \varphi(s_1)z - s_1\varphi(z) - \varphi(z)s_1 - z\varphi(s_1) - \varphi(t_1)z - t_1\varphi(z) - \varphi(z)t_1 - z\varphi(t_1) = 0$
$\Leftrightarrow \varphi(s_1+t_1)z + z\varphi(s_1+t_1) - \varphi(s_1)z - z\varphi(s_1) - \varphi(t_1)z - z\varphi(t_1) = 0$
$\Leftrightarrow [\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]z + z[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)] = 0.$
Therefore,
$[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{11}z = 0$
Based on condition i) in Definition 1, we obtain
$[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{11} = 0$
Therefore $[\varphi(s_1+t_1) - \varphi(s_1) - \varphi(t_1)]_{11} = 0$
Thus, it is proven that $\varphi(s_1+t_1) = \varphi(s_1) + \varphi(t_1)$.

(2) Take any $t_1 \in T_{11}, s_2, t_2 \in T_{22}$
It will be shown that $\varphi(s_2+t_2) = \varphi(s_2) + \varphi(t_2)$.
Furthermore,
Based on the mapping assumption $\varphi : T \longrightarrow T$, we obtain
$\varphi[(s_2+t_2)t_1 + t_1(s_2+t_2)]$
$= \varphi(s_2+t_2)t_1 + (s_2+t_2)\varphi(t_1) + \varphi(t_1)(s_2+t_2) + t_1\varphi(s_2+t_2)$
On the other hand,
$\varphi[(s_2+t_2)t_1 + t_1(s_2+t_2)] = 0$
$= \varphi(s_2t_1 + t_1s_2) + \varphi(t_2t_1 + t_1t_2)$

$$= \varphi(s_2)t_1 + s_2\varphi(t_1) + \varphi(t_1)s_2 + t_1\varphi(s_2) + \varphi(t_2)t_1 + t_2\varphi(t_1) + \varphi(t_1)t_2 + t_1\varphi(t_2).$$

From these two equations, we obtain

$$\Leftrightarrow \varphi(s_2 + t_2)t_1 + (s_2 + t_2)\varphi(t_1) + \varphi(t_1)(s_2 + t_2) + t_1\varphi(s_2 + t_2)$$
$$= \varphi(s_2)t_1 + s_2\varphi(t_1) + \varphi(t_1)s_2 + t_1\varphi(s_2) + \varphi(t_2)t_1 + t_2\varphi(t_1) + \varphi(t_1)t_2 + t_1\varphi(t_2)$$

$$\Leftrightarrow \varphi(s_2 + t_2)t_1 + (s_2 + t_2)\varphi(t_1) + \varphi(t_1)(s_2 + t_2) + t_1\varphi(s_2 + t_2) - \varphi(s_2)t_1 - s_2\varphi(t_1) - \varphi(t_1)s_2 - t_1\varphi(s_2) - \varphi(t_2)t_1 - t_2\varphi(t_1) - \varphi(t_1)t_2 - t_1\varphi(t_2) = 0$$

$$\Leftrightarrow \varphi(s_2 + t_2)t_1 + s_2\varphi(t_1) + t_2\varphi(t_1) + \varphi(t_1)s_2 + \varphi(t_1)t_2 + t_1\varphi(s_2 + t_2) - \varphi(s_2)t_1 - s_2\varphi(t_1) - \varphi(t_1)s_2 - t_1\varphi(s_2) - \varphi(t_2)t_1 - t_2\varphi(t_1) - \varphi(t_1)t_2 - t_1\varphi(t_2) = 0$$

$$\Leftrightarrow \varphi(s_2 + t_2)t_1 + t_1\varphi(s_2 + t_2) - \varphi(s_2)t_1 - t_1\varphi(s_2) - \varphi(t_2)t_1 - t_1\varphi(t_2) = 0$$

$$\Leftrightarrow [\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]t_1 + t_1[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)] = 0$$

Therefore

$$[\varphi(r_2 + s_2) - \varphi(r_2) - \varphi(s_2)]_{11}t_1 + t_1[\varphi(r_2 + s_2) - \varphi(r_2) - \varphi(s_2)]_{11} = 0$$

and

$$t_1[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]_{12} = 0$$

Based on conditions ii) and a) of Definition 1 and Theorem 1, we obtain

$[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]_{11} = 0$;

$[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]_{12} = 0$.

Next, it will be shown that $[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]_{22} = 0$

Take any $q \in T_{12}$

Based on the mapping assumption $\varphi : T \longrightarrow T$, we obtain

$$\varphi[(s_2 + t_2)q + q(s_2 + t_2)]$$
$$= \varphi(s_2 + t_2)q + (s_2 + t_2)\varphi(q) + \varphi(q)(s_2 + t_2) + q\varphi(s_2 + t_2)$$

On the other hand,

$$\begin{aligned} \varphi[(s_2 + t_2)q + q(s_2 + t_2)] &= \varphi(qs_2 + qt_2) \\ &= \varphi(qs_2) + \varphi(qt_2) \\ &= \varphi(qs_2 + s_2q) + \varphi(qt_2 + t_2q) \\ &= \varphi(q)s_2 + q\varphi(s_2) + \varphi(s_2)q + s_2\varphi(q) + \varphi(q)t_2 + q\varphi(t_2) + \varphi(t_2)q + t_2\varphi(q) \end{aligned}$$

From these two equations, we obtain

$$\Leftrightarrow \varphi(s_2 + t_2)q + (s_2 + t_2)\varphi(q) + \varphi(q)(s_2 + t_2) + q\varphi(s_2 + t_2)$$
$$= \varphi(q)s_2 + q\varphi(s_2) + \varphi(s_2)q + s_2\varphi(q) + \varphi(q)t_2 + q\varphi(t_2) + \varphi(t_2)q + t_2\varphi(q)$$

$$\Leftrightarrow \varphi(s_2 + t_2)q + (s_2 + t_2)\varphi(q) + \varphi(q)(s_2 + t_2) + q\varphi(s_2 + t_2) - \varphi(q)s_2 - q\varphi(s_2) - \varphi(s_2)q - s_2\varphi(q) - \varphi(q)t_2 - q\varphi(t_2) - \varphi(t_2)q - t_2\varphi(q) = 0$$

$$\Leftrightarrow \varphi(s_2 + t_2)q + s_2\varphi(q) + t_2\varphi(q) + \varphi(q)s_2 + \varphi(q)t_2 + q\varphi(s_2 + t_2) - \varphi(q)s_2 - q\varphi(s_2) - \varphi(s_2)q - s_2\varphi(q) - \varphi(q)t_2 - q\varphi(t_2) - \varphi(t_2)q - t_2\varphi(q) = 0$$

$$\Leftrightarrow \varphi(s_2 + t_2)q + q\varphi(s_2 + t_2) - q\varphi(s_2) - \varphi(s_2)q - q\varphi(t_2) - \varphi(t_2)q = 0$$

$$\Leftrightarrow [\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]q + q[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)] = 0$$

Therefore,

$$q[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]_{22} = 0$$

Based on condition ii) in Definition 1, we obtain

$$[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]_{22} = 0$$

Therefore $[\varphi(s_2 + t_2) - \varphi(s_2) - \varphi(t_2)]_{22} = 0$

Thus, it is proven that $\varphi(r_2 + s_2) = \varphi(r_2) + \varphi(s_2)$.

■

Lemma 6. *For all* $s_1 \in T_{11}, p \in T_{12}, s_2 \in T_{22}$ *holds*

$$\varphi(s_1 + p + s_2) = \varphi(s_1) + \varphi(p) + \varphi(s_2).$$

Proof.

Take any $s_1 \in T_{11}, p \in T_{12}, s_2 \in T_{22}$

It will be shown that $\varphi(s_1 + p + s_2) = \varphi(s_1) + \varphi(p) + \varphi(s_2)$.

Next,

For all $a_1 \in T_{11}$

$$\begin{aligned}\varphi[(s_1 + p + s_2)a_1 + a_1(s_1 + p + s_2)] &= \varphi(s_1 + p + s_2)a_1 + (s_1 + p + s_2)\varphi(a_1) + \varphi(a_1)(s_1 + p + s_2) \\ &\quad + a_1\varphi(s_1 + p + s_2)\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi[(s_1 + p + s_2)a_1 + a_1(s_1 + p + s_2)] &= \varphi(s_1a_1 + a_1s_1 + a_1p) \\ &= \varphi(s_1a_1 + a_1s_1) + \varphi(a_1p) \\ &= \varphi(s_1a_1 + a_1s_1) + \varphi(a_1p + pa_1) + \varphi(s_2a_1 + a_1s_2). \\ &= \varphi(s_1)a_1 + s_1\varphi(a_1) + \varphi(a_1)s_1 + a_1\varphi(s_1) + \varphi(a_1)p \\ &\quad + a_1\varphi(p) + \varphi(p)a_1 + p\varphi(a_1) + \varphi(s_2)a_1 \\ &\quad + s_2\varphi(a_1) + \varphi(a_1)s_2 + a_1\varphi(s_2)\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\Leftrightarrow \varphi(s_1 + p + s_2)a_1 + (s_1 + p + s_2)\varphi(a_1) + \varphi(a_1)(s_1 + p + s_2) + a_1\varphi(s_1 + p + s_2) \\ &\qquad = \varphi(s_1)a_1 + s_1\varphi(a_1) + \varphi(a_1)s_1 + a_1\varphi(s_1) + \varphi(a_1)p + a_1\varphi(p) + \varphi(p)a_1 \\ &\qquad + p\varphi(a_1) + \varphi(s_2)a_1 + s_2\varphi(a_1) + \varphi(a_1)s_2 + a_1\varphi(s_2) \\ &\Leftrightarrow \varphi(s_1 + p + s_2)a_1 + (s_1 + p + s_2)\varphi(a_1) + \varphi(a_1)(s_1 + p + s_2) + a_1\varphi(s_1 + p + s_2) \\ &\qquad - \varphi(s_1)a_1 - s_1\varphi(a_1) - \varphi(a_1)s_1 - a_1\varphi(s_1) - \varphi(a_1)p - a_1\varphi(p) - \varphi(p)a_1 \\ &\qquad - p\varphi(a_1) - \varphi(s_2)a_1 - s_2\varphi(a_1) - \varphi(a_1)s_2 - a_1\varphi(s_2) = 0 \\ &\Leftrightarrow \varphi(s_1 + p + s_2)a_1 + s_1\varphi(a_1) + p\varphi(a_1) + s_2\varphi(a_1) + \varphi(a_1)s_1 + \varphi(a_1)p + \varphi(a_1)s_2 \\ &\qquad + a_1\varphi(s_1 + p + s_2) - \varphi(s_1)a_1 - s_1\varphi(a_1) - \varphi(a_1)s_1 - a_1\varphi(s_1) - \varphi(a_1)p \\ &\qquad - a_1\varphi(p) - \varphi(p)a_1 - p\varphi(a_1) - \varphi(s_2)a_1 - s_2\varphi(a_1) - \varphi(a_1)s_2 - a_1\varphi(s_2) \\ &\qquad = 0 \\ &\Leftrightarrow \varphi(s_1 + p + s_2)a_1 + a_1\varphi(s_1 + p + s_2) - \varphi(s_1)a_1 - a_1\varphi(s_1) - a_1\varphi(p) - \varphi(p)a_1 \\ &\qquad - \varphi(s_2)a_1 - a_1\varphi(s_2) = 0 \\ &\Leftrightarrow [\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)]a_1 \\ &\qquad + a_1[\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)] = 0\end{aligned}$$

Therefore

$$\begin{aligned}&[\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)]_{11}a_1 \\ &\qquad + a_1[\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)]_{11} = 0\end{aligned}$$

and

$$a_1[\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)]_{12} = 0$$

Based on condition a) in Theorem 1 and condition ii) in Definition 1 obtained

$[\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)]_{11} = 0;$

$[\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)]_{12} = 0;$

Next, It will be shown $[\varphi(s_1 + p + s_2) - \varphi(s_1) - \varphi(p) - \varphi(s_2)]_{22} = 0$

Take any $z \in T_{12}$

Based on the mapping assumption $\varphi : T \longrightarrow T$

$$\begin{aligned}\varphi[(s_1 + p + s_2)z + z(s_1 + p + s_2)] &= \varphi(s_1 + p + s_2)z + (s_1 + p + s_2)\varphi(z) + \varphi(z)(s_1 + p + s_2) \\ &\quad + z\varphi(s_1 + p + s_2)\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi[(s_1+p+s_2)z+z(s_1+p+s_2)] &= \varphi(s_1z+pz+s_2z)\\ &= \varphi(s_1z)+\varphi(pz)+\varphi(s_2z)\\ &= \varphi(zs_1+s_1z)+\varphi(pz+zp)+\varphi(zs_2+s_2z)\\ &= \varphi(z)s_1+z\varphi(s_1)+\varphi(s_1)z+s_1\varphi(z)+\varphi(p)z+p\varphi(z)+\\ &\quad \varphi(z)p+z\varphi(p)+\varphi(z)s_2+z\varphi(s_2)+\varphi(s_2)z+s_2\varphi(z)\end{aligned}$$

From these two equations, we obtain

$$\begin{aligned}&\Leftrightarrow \varphi(s_1+p+s_2)z+(s_1+p+s_2)\varphi(z)+\varphi(z)(s_1+p+s_2)+z\varphi(s_1+p+s_2)\\ &\qquad = \varphi(z)s_1+z\varphi(s_1)+\varphi(s_1)z+s_1\varphi(z)+\varphi(p)z+p\varphi(z)+\varphi(z)p+z\varphi(p)\\ &\qquad +\varphi(z)s_2+z\varphi(s_2)+\varphi(s_2)z+s_2\varphi(z)\\ &\Leftrightarrow \varphi(s_1+p+s_2)z+(s_1+p+s_2)\varphi(z)+\varphi(z)(s_1+p+s_2)+z\varphi(s_1+p+s_2)-\varphi(z)s_1\\ &\qquad -z\varphi(s_1)-\varphi(s_1)z-s_1\varphi(z)-\varphi(p)z-p\varphi(z)-\varphi(z)p-z\varphi(p)-\varphi(z)s_2\\ &\qquad -z\varphi(s_2)-\varphi(s_2)z-s_2\varphi(z)=0\\ &\Leftrightarrow \varphi(s_1+p+s_2)z+s_1\varphi(z)+p\varphi(z)+s_2\varphi(z)+\varphi(z)s_1+\varphi(z)p+\varphi(z)s_2\\ &\qquad +z\varphi(s_1+p+s_2)-\varphi(z)s_1-z\varphi(s_1)-\varphi(s_1)z-s_1\varphi(z)-\varphi(p)z-p\varphi(z)\\ &\qquad -\varphi(z)p-z\varphi(p)-\varphi(z)s_2-z\varphi(s_2)-\varphi(s_2)z-s_2\varphi(z)=0\\ &\Leftrightarrow \varphi(s_1+p+s_2)z+z\varphi(s_1+p+s_2)-z\varphi(s_1)-\varphi(s_1)z-\varphi(p)z-z\varphi(p)-z\varphi(s_2)\\ &\qquad -\varphi(s_2)z=0\\ &\Leftrightarrow [\varphi(s_1+p+s_2)-\varphi(s_1)-\varphi(p)-\varphi(s_2)]z+z[\varphi(s_1+p+s_2)-\varphi(s_1)-\varphi(p)-\varphi(s_2)]\\ &\qquad =0\end{aligned}$$

Thus

$z[\varphi(s_1+p+s_2)-\varphi(s_1)-\varphi(p)-\varphi(s_2)]_{22}=0$

Based on condition i) in Definition 1, we obtain

$[\varphi(s_1+p+s_2)-\varphi(s_1)-\varphi(p)-\varphi(s_2)]_{22}=0;$

So, it is proven that $\varphi(r_1+m+r_2)=\varphi(r_1)+\varphi(m)+\varphi(r_2)$.

■

After discussing the lemmas needed to prove Theorem 1, we will prove Theorem 1.

Proof of Theorem 1.

Take any $a,b\in T$

Let $a=a_{11}+a_{12}+a_{22}$ and $b=b_{11}+b_{12}+b_{22}$.

We will show that if the mapping $\varphi: T\to T$ satisfies

$\varphi(ab+ba)=\varphi(a)b+a\varphi(b)+\varphi(b)a+b\varphi(a)$

For all $a,b\in T$, then the mapping φ additive. Furthermore, if T is 2-torsion free, then φ is a Jordan derivation.

$$\begin{aligned}\varphi(a+b) &= \varphi(a_{11}+a_{12}+a_{22}+b_{11}+b_{12}+b_{22})\\ &= \varphi[(a_{11}+b_{11})+(a_{12}+b_{12})+(a_{22}+b_{22})]\\ &= \varphi(a_{11}+b_{11})+\varphi(a_{12}+b_{12})+\varphi(a_{22}+b_{22})\\ &= \varphi(a_{11})+\varphi(b_{11})+\varphi(a_{12})+\varphi(b_{12})+\varphi(a_{22})+\varphi(b_{22})\\ &= \varphi(a_{11}+a_{12}+a_{22})+\varphi(b_{11}+b_{12}+b_{22})\\ &= \varphi(a)+\varphi(b)\end{aligned}$$

Since $\varphi(a+b)=\varphi(a)+\varphi(b)$, then φ additive.

Furthermore,

If T adalah 2-torsion free, then

$2\varphi(a^2)=\varphi(2a^2)=\varphi(aa+aa)=2[\varphi(a)a+a\varphi(a)]$.

Therefore, φ is a Jordan derivation.

■

4. CONCLUSION

Based on the results and discussions given in the previous chapter, it is obtained, for $\mathbf{2 \times 2}$ triangular matrix ring T with the following conditions:

1. If $r_1\Re_1 + \Re_1 r_1 = 0$, then $r_1 = 0$,
2. If $r_2\Re_2 + \Re_2 r_2 = 0$, then $r_2 = 0$.

If the mapping $\varphi : T \to T$ satisfies the multiplication derivation on the Jordan ring then φ is additive. By identifying these conditions, this research will contribute to a better understanding of $n \times n$ triangular matrix ring and probably can be solved with programming algorithms using software. Furthermore, if T is a 2-torsion-free ring then the additive mapping φ is a Jordan derivation. While this study focuses on triangular matrix rings, future work could explore the conditions for additivity in other non-associative structures, such as Lie algebras or alternative rings.

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