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Galois Group Correspondence On Extension Fields Over $\mathbb{Q}$

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Abstract. Let K/F be an extension field where $[K: F]$ denotes dimension of K as a vector space over F. Let $Aut(K/F)$ be the group of all automorphism of K that fixes F where the order of $Aut(K/F)$ is denoted by $|Aut(K/F)|$. Particularly, an extension field is called a Galois extension if $|Aut(K/F)| = [K: F]$. Moreover, we will give some properties of an extension field K/F which is a Galois extension. Using the properties of Galois extension, we will show that there is an one-one correspondence between the set of all intermediate fields in K and the set of all subgroups in $Aut(K/F)$. Furthermore, we will give some examples of Galois group correspondence using an extension field over $\mathbb{Q}$.

Keywords: Extension fields, Galois extension, Galois correspondence
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INTRODUCTION

Suppose F and K be fields where $F \subseteq K$. The field K is called an extension field of F and is denoted by K/F. We know that K can be viewed as a vector space over F. Thus, K have a basis where the dimension of K is denoted by $[K: F]$. Moreover, we form a set of all automorphisms of K that fixes F that is

$$Aut(K/F) = \{\sigma: K \to K \text{ automorphism} | \sigma(x) = x, for\ all\ x \in F\}$$

Note that $Aut(K/F)$ is a group under the operation of composition in $Aut(K/F)$. The group $Aut(K/F)$ is called automorphism group of K/F. The number of elements in $Aut(K/F)$ is called order of $Aut(K/F)$ and is written as $|Aut(K/F)|$. In particular, an extension field K/F is called a Galois extension K/F if $|Aut(K/F)| = [K: F]$.

Let K/F be an extension field with its automorphism group $G = Aut(K/F)$. An intermediate field E of K/F is a subfiend in K containing F that is $F \subseteq E \subseteq K$. Let H be a subgroup in G. Then, we form a set in K defined by

$$K^H = \{x \in K | \sigma(x) = x\ for\ every\ \sigma \in H\}.$$

In other words, K^H is the set of all elements in K which are mapped into itself by every $\sigma \in H$. The set K^H is a subfield in K containing F and is called fixed field of S. Thus, for every subgroup in G, we can form an intermediate subfield in K defined by K^H. Furthermore, suppose $\mathcal{H}$ is the set of all subgroups in G, and $\mathcal{F}$ is the set of all intermediate field of K/F. We can form a function between $\mathcal{H}$ and $\mathcal{F}$ defined by

$$\rho: \mathcal{H} \to \mathcal{F}$$
$$H \mapsto K^H$$

for all $H \in \mathcal{H}$. Using this correspondence, we can compute all subfields of K/F. For example, $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is an extension field where its automorphism group is $G = \{id, \sigma\}$ where $\sigma(1) = 1$ and $\sigma(\sqrt{2}) = -\sqrt{2}$. Note that, the set of all subgroups in G is $H_1 = \{id\}$ and $H_2 = G$ itself. Using the function, we obtain $\mathbb{Q}(\sqrt{2})^{H_1} = \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})^{H_2} = \mathbb{Q}$. Thus, the intermediate subfields of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}$.

Throughout this research, we will show that if K/F is a Galois extension then there is a one-one correspondence between the set of all subfields in K which contains F and the set of all subgroups in $Aut(K/F)$ (i.e. $\mathcal{F}$ and $\mathcal{H}$). We called this correspondence as Galois correspondence. Furthermore, we will give an example related to Galois group correspondence especially extension fields over $\mathbb{Q}$.

SOME RESULTS

In this part, we will discuss about an extension field K/F with its properties related to its role as a vector space over F. Next, we will also explain the automorphism group of an extension field K/F and give some examples on finding all automorphisms of K/F. Moreover, we will discuss about Galois extension with its properties. Using the properties of Galois extension, we will also discuss Galois corrrespondence.

Definition 1[3]
Let F and K be fields where $F \subseteq K$. The field K is called an extension field of F (denoted by K/F).

Example 2
i. $\mathbb{R}$ is an extension field of $\mathbb{Q}$.
ii. $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ is an extension field of $\mathbb{Q}$.
iii. $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = (Q(\sqrt{2})(\sqrt{3}) = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} | a, b, c, d \in \mathbb{Q}\}$ is an extension field of $\mathbb{Q}$.

Let K/F is an extension field. We know that K can be viewed as a vector space over F. Thus, K has a basis B over F where the number of elements in B is called dimension of K denoted by $[K: F]$.

Definition [3]
Let K/F is an extension field. If $[K: F] < \infty$ then K is called a **finite extension of F.**

Next, we will give an example of the dimension of a finite extension field.

Example 4
i. Given $\mathbb{Q}$ with its extension $\mathbb{Q}(\sqrt{2})$. Every $x \in \mathbb{Q}(\sqrt{2})$ can be expressed by
$$x = a + b\sqrt{2}.$$
Therefore, x can be written as a linear combination of $\{1, \sqrt{2}\}$. It is clear that $\{1, \sqrt{2}\}$ is linearly independent over $\mathbb{Q}$. So, $\{1, \sqrt{2}\}$ is a basis for $Q(\sqrt{2})$ over $\mathbb{Q}$. Hence, $[\mathbb{Q}(\sqrt{2}): \mathbb{Q}] = 2$.

ii. Let $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ be an extension field. Note that
$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = (\mathbb{Q}(\sqrt{2})(\sqrt{3}) = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} | a, b, c, d \in \mathbb{Q}\}.$$
Therefore, basis of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$ is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$. Thus, $[\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}] = 4$.

Suppose K/F is an extension field and E is a subfield in K containing F i.e. $F \subseteq E \subseteq K$. Thus, we obtain extension fields K/E and E/F. We will give a property of $[K: E]$ and $[E: F]$ in the following Lemma.

Lemma 5[3]
If K, E, F are fields where $F \subseteq E \subseteq K$ then $[K: F] = [K: E]. [E: F]$.
Proof
Let $[K: E] = m$ and $[E: F] = n$. We will show that $[K: F] = [K: E]. [E: F] = mn$.
Suppose that $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ be basis for K/E and E/F, respectively. Take any $x \in K$. Since K is a vector space over E, x can be expressed as
$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m.$$
for $\alpha_1, \alpha_2, \dots, \alpha_m \in E$. Note that E is a vector space over F, we obtain

$$\alpha_i = \beta_{i1}w_1 + \beta_{i2}w_2 + \cdots + \beta_{in}w_n$$

for $i = 1,2,\ldots,m$. Then,

$$x = (\beta_{11}w_1 + \beta_{12}w_2 + \cdots + \beta_{1n}w_n)v_1 + \cdots + (\beta_{m1}w_1 + \beta_{m2}w_2 + \cdots + \beta_{mn}w_n)v_m$$
$$= \beta_{11}v_1w_1 + \beta_{12}v_1w_2 + \cdots + \beta_{1n}v_1w_n + \cdots + \beta_{m1}v_mw_1 + \beta_{m2}v_mw_2 + \cdots + \beta_{mn}v_mw_n.$$

Thus, K is generated by $B = \{v_iw_j | i = 1,2,\ldots,m,\ j = 1,2,\ldots,n\}$. Now, we will show that B is linearly independent. Suppose that

$$c_{11}v_1w_1 + c_{12}v_1w_2 + \cdots + c_{1n}v_2w_n + \cdots + c_{m1}v_mw_1 + c_{m2}v_mw_2 + \cdots + c_{mn}v_mw_n = 0$$

So,

$$(c_{11}w_1 + c_{12}w_2 + \cdots + c_{1n}w_n)v_1 + \cdots + (c_{m1}w_1 + c_{m2}w_2 + \cdots + c_{mn}w_n)v_m = 0.$$

Since $\{v_1, v_2, \ldots, v_m\}$ is linearly independent, we obtain $c_{i1}w_1 + c_{i2}w_2 + \cdots + c_{in}w_n = 0$ for $i = 1,2,\ldots,m$. Also, since $\{w_1, w_2, \ldots, w_n\}$ is linearly independent, it means $c_{i1} = c_{i2} = \cdots = c_{in} = 0$. Thus, $c_{ij} = 0$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. We have B is a basis of K over F. Hence, $B = \{v_iw_j | i = 1,2,\ldots,m,\ j = 1,2,\ldots,n\}$ and $[K:F] = mn$.
■

Next, we will discuss automorphism group of an extension field. Moreover, we will give some properties related to the automorphism group.

Let K/F be an extension field. We form the set of all automorphism of K which is defined by

$$Aut(K/F) = \{\sigma: K \to K\ automorphism\ | \sigma(x) = x\ , for\ all\ x \in F\ \}.$$

$Aut(K/F)$ is a group under the operation of composition and is called **the automorphism group of K/F.**

Next, we will give some examples of $Aut(K/F)$ of extension field K/F.

Example 6

Suppose an extension field $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ with its basis $B = \{1, \sqrt{2}\}$. It is known that each automorphism can be defined by a function

$$\rho: B \to \mathbb{Q}(\sqrt{2}).$$

The function will then be extended to $\rho': \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$. Because σ is an element in $Aut(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, we have $\sigma(1) = 1$ and $\sigma(a) = \sigma(1.a) = a.\sigma(1) = a.1 = a$ for every $a \in \mathbb{Q}$. Note that,

$$0 = \sigma(0) = \sigma\left((\sqrt{2})^2 - 2\right) = \sigma(\sqrt{2})^2 - 2.$$

So, $\sigma(\sqrt{2})^2 = 2$ and $\sigma(\sqrt{2}) = \sqrt{2}$ or $-\sqrt{2}$. So, we get two automorphisms of $\mathbb{Q}(\sqrt{2})$ which is defined by

$$\sigma_1: B \to \mathbb{Q}(\sqrt{2})$$
$$1 \mapsto 1$$
$$\sqrt{2} \mapsto \sqrt{2}$$

and

$$\sigma_2: B \to \mathbb{Q}(\sqrt{2})$$
$$1 \mapsto 1$$
$$\sqrt{2} \mapsto -\sqrt{2}.$$

Then, those two functions are extended to

$$\sigma_1': \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$$
$$a.1 + b.\sqrt{2} \mapsto a.\sigma_1(1) + b.\sigma_1(\sqrt{2})$$

and

$$\sigma_2': \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$$
$$a.1 + b.\sqrt{2} \mapsto a.\sigma_1(1) + b.\sigma_1(-\sqrt{2})$$

Therefore, $Aut(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{\sigma_1', \sigma_2'\} = \{id, \sigma_2'\}$. Thus, we have extension field $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ with its automorphism group $G = Aut(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{id, \sigma_2'\}$.

Example 7

Given an extension field $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ where

$$\mathbb{Q}(\sqrt[3]{2}) = \{a.1 + b.\sqrt[3]{2} + c.\sqrt[3]{4}\}.$$

So, $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is a basis of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$. We will use the same way from **Example 6** to find all automorphisms of $\mathbb{Q}(\sqrt[3]{2})$. We construct all automorphisms in $\mathbb{Q}(\sqrt[3]{2})$ from bijective function which is defined by

$$\rho: B \to \mathbb{Q}(\sqrt[3]{2}).$$

We obtain $\sigma(1) = 1$ and $\sigma(a) = \sigma(1.a) = a.\sigma(1) = a.1 = a$ for every $a \in \mathbb{Q}$. So,

$$0 = \sigma(0) = \sigma((\sqrt[3]{2})^3 - 2) = \sigma((\sqrt[3]{2}))^3 - \sigma(2) = \sigma(\sqrt[3]{2})^3 - 2.$$

So,

$$\sigma(\sqrt[3]{2})^3 = 2.$$

We know that the roots of $x^3 - 2 = 0$ are $\sqrt[3]{2}\, e^{\frac{1}{3}.2\pi i}\sqrt[3]{2}, \sqrt[3]{2}\, e^{\frac{2}{3}.2\pi i}$,and$\sqrt[3]{2}$. Note that $\sqrt[3]{2}\, e^{\frac{1}{3}.2\pi i}\sqrt[3]{2}, \sqrt[3]{2}\, e^{\frac{2}{3}.2\pi i} \notin \mathbb{Q}(\sqrt[3]{2})$, so $\sigma(\sqrt[3]{2}) = \sqrt[3]{2}$. Using the same way, we will also only have $\sigma(\sqrt[3]{4}) = \sqrt[3]{4}$. Hence, we can only form one automorphism defined by

$$\begin{gathered} \sigma_1: B \to \mathbb{Q}(\sqrt[3]{2}) \\ 1 \mapsto 1 \\ \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ \sqrt[3]{4} \mapsto \sqrt[3]{4} \end{gathered}$$

Then, we extend σ_1 to $\sigma_1{}'$ defined by

$$\begin{gathered} \sigma_1{}': \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{Q}(\sqrt[3]{2}) \\ a.1 + b.\sqrt[3]{2} + c.\sqrt[3]{4} \mapsto a.\sigma_1(1) + b.\sigma_1(\sqrt[3]{2}) + c.\sigma_1(\sqrt[3]{4}) \\ a.1 + b.\sqrt[3]{2} + c.\sqrt[3]{4} \mapsto a.1 + b.\sqrt[3]{2}\, c + \sqrt[3]{4}. \end{gathered}$$

Thus, $\sigma_1{}'$ is the identity function of $\mathbb{Q}(\sqrt[3]{2})$. In conclusion, we obtain $Aut(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{\sigma_1{}'\} = \{id\}$.

Example 8

Suppose an extension field $\mathbb{Q}(\sqrt{2}, \sqrt{3})/Q$ with its basis $B = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$. It is known that each automorphism can be defined by a function

$$\sigma: B \to \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

The function will then be extended to $\sigma': \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$. Because $\sigma \in Aut(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, we have $\sigma(1) = 1$ because $\sigma(a) = a$ for every $a \in \mathbb{Q}$. Note that,

$$0 = \sigma(0) = \sigma\left((\sqrt{2})^2 - 2\right) = \sigma(\sqrt{2})^2 - 2,$$

$$0 = \sigma(0) = \sigma\left((\sqrt{3})^2 - 3\right) = \sigma(\sqrt{3})^2 - 3$$

So, $\sigma(\sqrt{2})^2 = 2$ and $\sigma(\sqrt{2}) = \sqrt{2}$ or $-\sqrt{2}$. Also, $\sigma(\sqrt{3})^2 = 3$ so that $\sigma(\sqrt{3}) = 3$ or $-\sqrt{3}$. Note that $\sigma(\sqrt{6}) = \sigma(\sqrt{2})\sigma(\sqrt{3})$. It means $\sigma(\sqrt{6})$ depends on $\sigma(3)$ and $\sigma(\sqrt{3})$. So, we get four automorphisms of $Q(\sqrt{2})$ which is defined by

$\sigma_1: B \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$	$\sigma_2: B \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$	$\sigma_3: B \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$	$\sigma_4: B \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$
$1 \mapsto 1$	$1 \mapsto 1$	$1 \mapsto 1$	$1 \mapsto 1$
$\sqrt{2} \mapsto \sqrt{2}$	$\sqrt{2} \mapsto -\sqrt{2}$	$\sqrt{2} \mapsto \sqrt{2}$	$\sqrt{2} \mapsto -\sqrt{2}$
$\sqrt{3} \mapsto \sqrt{3}$	$\sqrt{3} \mapsto \sqrt{3}$	$\sqrt{3} \mapsto -\sqrt{3}$	$\sqrt{3} \mapsto -\sqrt{3}$
$\sqrt{6} \mapsto \sqrt{6}$	$\sqrt{6} \mapsto -\sqrt{6}$	$\sqrt{6} \mapsto -\sqrt{6}$	$\sqrt{6} \mapsto \sqrt{6}$

Next, we extend those four automorphisms to $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ defined by

$$\begin{gathered} \sigma_i{}': Q(\sqrt{2}, \sqrt{3}) \to Q(\sqrt{2}, \sqrt{3}) \\ a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6} \mapsto a.\sigma_i(1) + b.\sigma_i(\sqrt{2}) + c.\sigma_i(\sqrt{3}) + d.\sigma_i(\sqrt{6}) \end{gathered}$$

Thus, $Aut(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) = \{\sigma_1', \sigma_2', \sigma_3', \sigma_4'\}$. Note that $\sigma_1' = id$ and $\sigma_4' = \sigma_2'\sigma_3'$. Hence, $Aut(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) = \{id, \sigma_2', \sigma_3', \sigma_2'\sigma_3'\}$.

Next, we will give a property of $Aut(K/F)$ in this following lemma.

Proposition 9[5]
If $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is the set of automorphisms of K then $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is linearly independent (i.e. if $\alpha_1\sigma_1 + \alpha_2\sigma_2 + \cdots + \alpha_n\sigma_n = 0$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$).
Proof.
Suppose that $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is the set of automorphisms of K. We will prove that $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is linearly independent using induction method on k elements of the given set.

i. For $k = 1$. We take any σ_i for $i = 1,2, \dots, n$ where $\alpha_i\sigma_i = 0$. It means $(\alpha_1\sigma_1)(x) = \alpha_1(\sigma_1(x)) = 0$. Note that K is a field and σ_i is an automorphism, then we have $\sigma_1(x) \neq 0$ for every nonzero $x \in K$. Therefore, $\alpha_i = 0$.
ii. It holds for k where $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$ is linearly independent.
iii. We will prove that also holds for $k + 1$. Suppose that

$$\alpha_1\sigma_1 + \alpha_2\sigma_2 + \cdots + \alpha_{k+1}\sigma_{k+1} = 0$$

where $\alpha_1, \alpha_2, \dots, \alpha_{k+1} \in F$. So, for every $x \in K$

$$(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \cdots + \alpha_{k+1}\sigma_{k+1})(x) = 0.$$

Thus,

$$\alpha_1\sigma_1(x) + \alpha_2\sigma_2(x) + \cdots + \alpha_{k+1}\sigma_{k+1}(x) = 0. \qquad \text{(i)}$$

Because $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ are distinct, there is a nonzero $y \in K$ such that $\sigma_1(y) \neq \sigma_2(y)$. Using equation (i), we obtain

$$\Leftrightarrow \alpha_1\sigma_1(xy) + \alpha_2\sigma_2(xy) + \cdots + \alpha_{k+1}\sigma_{k+1}(xy) = 0$$
$$\Leftrightarrow \alpha_1\sigma_1(x)\sigma_1(y) + \alpha_2\sigma_2(x)\sigma_2(y) + \cdots + \alpha_{k+1}\sigma_{k+1}(x)\sigma_{k+1}(y) = 0 \qquad \text{(ii)}$$

From (i), we obtain

$$\alpha_1\sigma_1(x) = -\alpha_2\sigma_2(x) - \cdots - \alpha_{k+1}\sigma_{k+1}(x) \qquad \text{(iii)}$$

Then, we substitute (iii) to (ii)

$$\Leftrightarrow (-\alpha_2\sigma_2(x) - \alpha_3\sigma_3(x) - \cdots - \alpha_{k+1}\sigma_{k+1}(x))\sigma_1(y) + \alpha_2\sigma_2(x)\sigma_2(y) + \cdots + \alpha_{k+1}\sigma_{k+1}(x)\sigma_{k+1}(y) = 0$$
$$\Leftrightarrow -\alpha_2\sigma_2(x)\sigma_1(y) - \alpha_3\sigma_3(x)\sigma_1(y) \dots - \alpha_{k+1}\sigma_{k+1}(x)\sigma_1(y) + \alpha_2\sigma_2(x)\sigma_2(y) + \cdots + \alpha_{k+1}\sigma_{k+1}(x)\sigma_{k+1}(y) = 0$$
$$\Leftrightarrow -\alpha_2\sigma_2(x)\sigma_1(y) - \alpha_3\sigma_3(x)\sigma_1(y) - \cdots - \alpha_{k+1}\sigma_{k+1}(x)\sigma_1(y) + \alpha_2\sigma_2(x)\sigma_2(y) + \alpha_3\sigma_3(x)\sigma_3(y) + \cdots + \alpha_{k+1}\sigma_{k+1}(x)\sigma_{k+1}(y) = 0$$
$$\Leftrightarrow \alpha_2\sigma_2(x)\big(\sigma_2(y) - \sigma_1(y)\big) + \alpha_3\sigma_3(x)\big(\sigma_3(y) - \sigma_1(y)\big) \dots + \alpha_{k+1}\sigma_{k+1}(x)\big(\sigma_{k+1}(y) - \sigma_1(y)\big) = 0$$
$$\Leftrightarrow \alpha_2\big(\sigma_2(y) - \sigma_1(y)\big)\sigma_2(x) + \alpha_3\big(\sigma_3(y) - \sigma_1(y)\big)\sigma_3(x) + \cdots + \alpha_{k+1}\big(\sigma_{k+1}(y) - \sigma_1(y)\big)\sigma_{k+1}(x) = 0$$
$$\Leftrightarrow (\alpha_2\big(\sigma_2(y) - \sigma_1(y)\big)\sigma_2 + \alpha_3\big(\sigma_3(y) - \sigma_1(y)\big)\sigma_3 \dots + \alpha_{k+1}\big(\sigma_{k+1}(y) - \sigma_1(y)\big)\sigma_{k+1})\,(x) = 0$$

Using the assumption for k, we obtain
$\alpha_2\big(\sigma_2(y) - \sigma_1(y)\big) = \alpha_2\big(\sigma_2(y) - \sigma_1(y)\big) = \cdots = \alpha_{k+1}\big(\sigma_{k+1}(y) - \sigma_1(y)\big) = 0.$

Note that $\alpha_2\big(\sigma_2(y) - \sigma_1(y)\big) = 0$ and $(y) \neq \sigma_1(y)$, so we have $\alpha_2 = 0$. Moreover, using (i) and $\alpha_2 = 0$, we also have

$$\Leftrightarrow \alpha_1\sigma_1(x) + \alpha_3\sigma_3(x) \dots + \alpha_{k+1}\sigma_{k+1}(x) = 0$$
$$\Leftrightarrow (\alpha_1\sigma_1 + \alpha_3\sigma_3 + \cdots + \alpha_{k+1}\sigma_{k+1})(x) = 0.$$

Therefore, $\alpha_1\sigma_1 + \alpha_3\sigma_3 + \cdots + \alpha_{k+1}\sigma_{k+1} = 0$. Again, using the assumption for $n = k$, it implies that that $\alpha_1 = \alpha_3 = \cdots = \alpha_{k+1} = 0$. Hence, $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is linealy independent over F. ∎

Moreover, we will give the relation between $|Aut(K/F)|$ and $[K: F]$ in the proposition below.

Proposition 10 [5]
If K/F is an extension field then $|Aut(K/F)| \leq [K: F]$.

Proof

Write $G = Aut(K/F)$. Suppose $G = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ so that $|G| = n$. Let $[K:F] = n$ and the basis of K/F is $B = \{v_1, v_2, \ldots, v_d\}$ for some $d \in \mathbb{N}$. We will prove that $n \leq d$ using method of contradiction.
Suppose $n > d$. We form a linear equation system i.e.

$$\begin{gathered}\sigma_1(v_1)x_1 + \sigma_2(v_1)x_2 + \cdots + \sigma_n(v_1)x_n = 0\\ \sigma_1(v_2)x_1 + \sigma_2(v_2)x_2 + \cdots + \sigma_n(v_2)x_n = 0\\ \vdots\\ \sigma_1(v_d)x_1 + \sigma_2(v_d)x_2 + \cdots + \sigma_n(v_d)x_n = 0.\end{gathered}$$

Note that there are more variables than the number of equations. It implies there is a nonzero solution, $\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}$ where $c_i \neq 0$ for some $i \in \{1,2,\ldots,n\}$. Let $w \in K/F$. It means w can be expressed as

$$w = a_1v_1 + a_2v_2 + \cdots + a_dv_d$$

where $a_1, a_2, \ldots, a_d \in F$. Then, we multiply a_i to the system of equations. Thus,

$$\begin{gathered}a_1\sigma_1(v_1)x_1 + a_1\sigma_2(v_1)x_2 + \cdots + a_1\sigma_n(v_1)x_n = 0\\ a_2\sigma_1(v_2)x_1 + a_2\sigma_2(v_2)x_2 + \cdots + a_2\sigma_n(v_2)x_n = 0\\ \vdots\\ a_d\sigma_1(v_d)x_1 + a_d\sigma_2(v_d)x_2 + \cdots + a_d\sigma_n(v_d)x_n = 0.\end{gathered}$$

Therefore,

$$\begin{aligned}(a_1\sigma_1(v_1) + a_2\sigma_1(v_2) + \cdots + a_d\sigma_1(v_d))c_1 + (a_1\sigma_2(v_1) + a_2\sigma_2(v_2) + \cdots + a_d\sigma_2(v_d))c_2 + \cdots + (a_1\sigma_n(v_1)\\ + a_2\sigma_n(v_2) + \cdots + a_d\sigma_n(v_d))c_n = 0\end{aligned}$$

and

$$\begin{aligned}\sigma_1(a_1v_1 + a_2v_2 + \cdots + a_dv_d).c_1 + \sigma_2(a_1v_1 + a_2v_2 + \cdots + a_dv_d).c_2 + \cdots + \sigma_n(a_1v_1 + a_2v_2 + \cdots + a_dv_d).c_n\\ = 0.\end{aligned}$$

So, $c_1.\sigma_1(w) + c_2.\sigma_2(w) + \cdots + c_n\sigma_n(w) = 0$ and $(c_1\sigma_1 + c_1\sigma_2 + \cdots + c_n\sigma_n)(w) = 0$. It holds for every $w \in K/F$. It implies that $\alpha_1\sigma_1 + \alpha_2\sigma_2 + \cdots + \alpha_n\sigma_d = 0$. Note that there is $c_i \neq 0$ for some $i = 1,2,\ldots,n$. Hence, $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is linearly independent. It implies contradiction with **Proposition 7**. Hence, $n \leq d$ that is $|G| \leq [K:F]$. ∎

Based on **Proposition 10**, we have $|Aut(K/F)| \leq [K:F]$. However, the equality does not always hold to all extension fields. We will give an example to describe it.

Example 11

Given an extension field $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. From **Example 4**, we know that $\mathbb{Q}(\sqrt[3]{2}) = \{a.1 + b.\sqrt[3]{2} + c.\sqrt[3]{4}\}$ So, $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is a basis of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$. We also have $Aut(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{id\}$. Thus, $[\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}] = 3$ and $|Aut(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})| = 1$.

Based on the example above, it then motivates the definition of Galois extension. We will give the definition of Galois extension on the following definition.

Definition 12[5]

Let K/F be a finite extension field. K is called Galois extension over F if $|Aut(K/F)| = [K:F]$.

It's common to write the automorphism $Aut(K/F)$ as $\boldsymbol{Gal(K/F)}$ when K is a Galois extension and is called Galois group of K/F. Next, we will give example of a Galois extension and a non-Galois extension in the following example.

Example 13

i. Using **Example 6**, we have $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a Galois extension. Because the basis of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is$\{1, \sqrt{2}\}$. We obtain $Aut(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{id, \sigma_2\}$. Thus, $|Aut(\mathbb{Q}(\sqrt{2})/\mathbb{Q})| = [\mathbb{Q}(\sqrt{2}): \mathbb{Q}] = 2$. Hence, $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a Galois extension field over $\mathbb{Q}$.

ii. Based on **Example 7**, we know that $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois extension because $Aut(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{id\}$ and the basis of $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is $\{1, \sqrt[3]{2}\}$. So, $|Aut(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})| \neq [\mathbb{Q}(\sqrt[3]{2}): \mathbb{Q}] = 2$.

Let K/F be an extension field and $Aut(K/F)$ be the automorphism group of K/F. For every, $S \subseteq Aut(K/F)$, We form a subset of K defined by

$$K^S = \{x \in K | \sigma(x) = x, \forall \sigma \in S\}.$$

Note that $\forall a, b \in K^S$ dan $\sigma \in S$, we obtain

$$\sigma(a - b) = \sigma(a) - \sigma(b) = a - b$$

and

$\sigma(ab^{-1}) = \sigma(a)\sigma(b^{-1}) = \sigma(a)(\sigma(b))^{-1} = ab^{-1}$.

Therefore, K^S is a subfield in K containing F and is called **the fixed field of S** [5]. In other words, **S fixed all elements** in K^S.

Example 14

Using **Example 6**, we have $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. We obtain $G = Aut(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{id, \sigma_2{'}\}$ where

$$id: \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$$
$$a.1 + b.\sqrt{2} \mapsto a.\sigma_1(1) + b.\sigma_1(\sqrt{2})$$

and

$$\sigma_2{'}: \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$$
$$a.1 + b.\sqrt{2} \mapsto a.\sigma_1(1) + b.\sigma_1(-\sqrt{2}).$$

Thus, $id(a.1) = a$ and $\sigma_2'(a.1) = a$ where $a \in \mathbb{Q}$. Hence, $\mathbb{Q}(\sqrt{2})^G = \mathbb{Q}$.

Let K/F be an extension field where it automorphism group is $G = Aut(K/F)$. Suppose H is a subgroup in H. Next, we will give a property related to fixed field of a H which is denoted by K^H in this following Lemma.

Theorem 15 [5]

Let K/F be an extension field where $[K:F] < \infty$. If $K^G = F$ then $[K:F] = |Aut(K/F)|$.

Proof.

Let $[K:F] = d$ and $|Aut(K/F)| = n$. Based on **Proposition 10**, we have $d \geq n$. Next, we will prove that $d \leq n$ using method of contradiction.

Suppose $d > n$. Thus, there exist $n + 1$ elements $v_1, v_2, \dots, v_{n+1}$ which are linearly independent over F. Then, we construct the following system of equations

$$\begin{aligned} \sigma_1(v_1)x_1 + \sigma_1(v_2)x_2 + \cdots + \sigma_1(v_{n+1})x_{n+1} = 0 \\ \sigma_2(v_1)x_1 + \sigma_2(v_2)x_2 + \cdots + \sigma_2(v_{n+1})x_{n+1} = 0 \\ \vdots \\ \sigma_n(v_1)x_1 + \sigma_2(v_2)x_2 + \cdots + \sigma_n(v_{n+1})x_{n+1} = 0. \end{aligned}$$

Note that there are more variables than the number of equations. It implies there is a non-trivial solution, $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n+1} \end{pmatrix}$ where $\alpha_i \neq 0$ for some $i \in \{1,2, \dots, n+1\}$. Among all non-trivial solutions, we choose r as the least number of nonzero elements. Moreover, $r \neq 1$ because $\sigma_1(v_1)\alpha_1 = 0$ implies $\sigma_1(v_1) = 0$ and $v_1 = 0$.

i. We will prove that there exists a non-trivial solutions where α_i are in F for any $i \in \{1,2,\dots,n+1\}$. Supposed $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a non-trivial solution with r non-zero elements where $\alpha_1, \alpha_2, \dots, \alpha_r \neq 0$. We obtain a new non-trivial solution by multiplying the given solution with $\frac{1}{\alpha_r}$ which is $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1/\alpha_r \\ \alpha_2/\alpha_r \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Thus,

$$\beta_1 \sigma_i(v_1) + \beta_2 \sigma_i(v_2) + \cdots + 1.\sigma_i(v_{n+1}) = 0 \qquad (*)$$

For $i = 1,2,\dots,n$. Now, we will show that β_i are in F for any $i \in \{1,2,\dots,n+1\}$ using method of contradiction. Suppose there exists $\beta_i \notin F$, say β_1. We know that $F = K^G$ so that β_1 is not an element of the fixed field. In other words, there exists $\sigma_k \in G$ where $\sigma_k(\beta_1) \neq \beta_1$. So, $\sigma_k(\beta_1) - \beta_1 \neq 0$. Since G is a group, it implies $\sigma_k G = G$. It means for any $\sigma_i \in G$, we obtain $\sigma_i = \sigma_k \sigma_j$ for $j = 1,2,\dots,n$. Applying σ_k to the expressions of (*)

$$\Leftrightarrow \sigma_k(\beta_1 \sigma_j(v_1) + \beta_2 \sigma_j(v_2) + \cdots + 1.\sigma_j(v_r)) = 0$$
$$\Leftrightarrow \sigma_k(\beta_1).\sigma_k\sigma_j(v_1) + \sigma_k(\beta_2).\sigma_k\sigma_j(v_2) + \cdots + \sigma_k\sigma_j(v_r) = 0$$

for $j = 1,2,\dots,n$ so that from $\sigma_i = \sigma_k\sigma_j$. We obtain

$$\sigma_k(\beta_1).\sigma_i(v_1) + \sigma_k(\beta_2).\sigma_i(v_2) + \cdots + \sigma_i(v_r) = 0. \qquad (**)$$

Subtracting (*) and (**), we have

$$(\beta_1 - \sigma_k(\beta_1)\sigma_i(v_1) + (\beta_2 - \sigma_k(\beta_2)\sigma_i(v_2) + \cdots + (\beta_{r-1} - \sigma_k(\beta_{r-1})\sigma_i(v_{r-1}) + 0 = 0$$

which is non-trivial solution because $\sigma_k(\beta_1) \neq \beta_1$ and is having $r-1$ non-zeo elements, contrary to the choice of r as the minimality. Hence, $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a non-trivial where all $\beta_i \in F$ for any $i = 1,2,\dots,n$.

ii. Using (i), we obtain a nonzero solution with all elements are in F. So, using the first equation in the system, we obtain

$$\sigma_1(v_1)\beta_1 + \sigma_1(v_2)\beta_2 + \cdots + \sigma_1(v_r)\beta_r = 0$$
$$\sigma_1(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_r v_r) = 0.$$

Because $\sigma_1$is an automorphism, we obtain $\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_r v_r = 0$ where $\beta_1, \beta_2, \dots, \beta_r$ are nonzero elements in K. It is contrary to $v_1, v_2, \dots, v_{n+1}$ which are linearly independent over F.

Thus, we have $d \leq n$. Hence, $d = n$ i.e. $[K:F] = |Aut(K/F)|$. ■

Next, we will give a neccesary and sufficient contdition for K/F is Galois using its fixed field.

Corollary 16[5]
Let K/F be an extension field where $[K:F] < \infty$ with its automorphism group $G = Aut(K/F)$. The field K/F is a Galois extension over F if and only if $K^G = F$.
Proof.
($\Rightarrow$) We have K is a Galois extension over F. It means $[K:F] = |Aut(K/F)|$. We will show that $K^G = F$. We know that K^G is a subfield of K and $F \subseteq K^G \subseteq K$. Based on **Lemma 5** and **Theorem 15,** we obtain

$$|Aut(K/F)| = [K:K^G] = [K:F]/[K^G:F].$$

Because $[K:F] = |Aut(K/F)|$. It implies $[K^G:F] = 1$. Hence, $K^G = F$.

($\Leftarrow$) We know that $K^G = F$. Using **Theorem 15**, we have $[K:K^G] = [K:F] = |Aut(K/F)|$. Thus, K is a Galois extension over F. ■

Let K/F be an extension field with its automorphsim group $G = Aut(K/F)$. Using the Corollary above, we can determine that K/F is a Galois extension by showing that the fixed field of its automorphism group G is F itself (that is $K^G = F$).

Lemma 17 [5]
Let K/F be an extension field and E be an intermediate field of K/F that is $F \subseteq E \subseteq K$. The automorphism group $Aut(K/E)$ is a subgroup in $Aut(K/F)$.
Proof.
Let K/F be an extension field and E be an intermediate field of K/F. Write $G = Aut(K/F)$. Note that K/E is an extension field. So, $H = Aut(K/E)$ is the automorphism group of K/E where

$$Aut(K/E) = \{\sigma: K \to K \text{ automorphism } | \sigma(x) = x, for\ all\ x \in E\ \}.$$

Moreover, let $\sigma \in H$. It means, $\sigma(x) = x$ for all $x \in E$. Because $F \subseteq E$, so $\sigma(x) = x$ for all $x \in F \subseteq E$. Thus, $\sigma \in Aut(K/F) = G$. Hence, H is group and a subset in G. It implies that H is a subgroup of G. ■

Lemma 18 [5]
Let K/F be Galois extension field. If E is an intermediate field of K/F then K/E is a Galois extension.
Proof.
Let K/F be Galois extension field. If E is an intermediate field of K/F. We have, K/E is an extension field with it automorphism group $H = Aut(K/E)$. Based on **Corollary 16**, we will prove that K/E is a Galois extension by showing that E is the fixed field of its automorphism group $Aut(K/E)$ i.e. $E = K^{Aut(K/E)}$. Write $G = Aut(K/F)$.

Suppose H is a subgroup of G where its fixed field is E i.e. $E = K^H$.

i. First, we will show that $H = Aut(K/E)$. Let $\sigma \in H \subseteq G$. We know that H fixes all element in E. So,

$$\sigma(x) = x$$

for all $x \in E$. Using the definition of $Aut(K/E)$, we have $\sigma \in Aut(K/E)$. Thus, $H \subseteq Aut(K/E)$ and $|H| \leq |Aut(K/K^H)|$. Based on **Theorem 15** , we have

$$[K:K^H] = |H|.$$

Note that K/K^H is an extension field, so $|Aut(K/K^H)| \leq [K:K^H]$ based on **Proposition 10**. Therefore,

$$|H| \leq |Aut(K/K^H)| \leq [K:K^H] = |H|.$$

Thus, $|H| = |Aut(K/K^H)|$. Because $|H|$ and $|Aut(K/K^H)|$ are finite and also $H \subseteq Aut(K/E)$, it implies $H = Aut(K/K^H) = Aut(K/E)$. In other words, E is the fixed field of $Aut(K/E)$.

ii. We have E is the fixed field of $Aut(K/E)$ from (i). It means, $E = K^{Aut(K/E)}$. Using **Corollary 16**, we have K/E is a Galois extension with Galois group $H = Aut(K/K^H) = Aut(K/E)$. ■

Let K/F be a Galois extension field where $Aut(K/F)$ is the automorphism group of K/F. We know that for all subgroups in G, we can form an intermediate subfield in K. Suppose

$\mathcal{H}$ is the set of all subgroups in G, and
$\mathcal{F}$ is the set of all intermediate field of K/F.

We can form a function between $\mathcal{H}$ and $\mathcal{F}$ defined by

$$\rho: \mathcal{H} \to \mathcal{F}$$
$$H \mapsto K^H$$

for all $H \in \mathcal{H}$. In other words, H is mapped to its fixed field K^H. Using the property of K/F as a Galois extension, we will show that there is a one-one correspondence between $\mathcal{H}$ and $\mathcal{F}$ that is ρ is bijective.

Theorem 19[5]
Let K/F be an extension field. If K is a Galois extension then there is an one-one correspondence between intermediate field E of K/F and subgroups H of G defined by

$$\rho: \mathcal{H} \to \mathcal{F}$$
$$H \mapsto K^H.$$

Proof
Let K/F be a Galois extension field where $Aut(K/F)$ is the automorphism group of K/F. we will show that there is a one-one correspondence between $\mathcal{H}$ and $\mathcal{F}$ that is ρ is bijective.

i. Suppose E is an intermediate field. From **Lemma 18**, we have K/E is a Galois extension with its Galois group $H = Aut(K/E)$. We know that H is a subgroup in G. Thus, E is the fixed field of H that is $E = K^H = \rho(H)$. Hence, ρ is surjective.
ii. Let $H_1, H_2 \in \mathcal{H}$ where G where $\rho(H_1) = \rho(H_2)$ that is $K^{H_1} = K^{H_2}$. Note that K/K^{H_1} and K/K^{H_2} are Galois extensions by **Lemma 18**. So, $H_1 = Aut(K/K^{H_1})$ and $H_2 = Aut(K/K^{H_2})$. Also, note that $K^{H_1} = K^{H_2}$ so that K^{H_1} is the fixed field of H_2. Thus, $H_2 \subseteq Aut(K/K^{H_1}) = H_1$. Analogously, $K^{H_2} = K^{H_1}$. We have, K^{H_2} is the fixed field of H_1. Hence, $H_1 \subseteq Aut(K/K^{H_2}) = H_2$. Therefore, $H_1 = H_2$. Hence, ρ is injective

From (i) and (ii), it implies that, ρ is bijective so that there is an one-one correspondence between set of all subgroups in G and the set of all intermediate field of K/F. ■

Next, we will describe the Galois correspondence using Galois extension field $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ in this following example.

Example 20
Using **Example 8**, we have $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension where its basis $B = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ and $G = Aut(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) = \{ id, \sigma_2', \sigma_3', \sigma_2'\sigma_3'\}$. Note that $Aut(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Klein group generated by $\{\sigma_2', \sigma_3'\}$. Next, we will find all intermediate fields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ using the Galois correspondence. Since, G is a Klein group, we can compute all subgroups in G which are

$$H_1 = \{id\} \qquad H_2 = \{id, \sigma_2'\} \qquad H_3 = \{id, \sigma_3'\} \qquad H_4 = \{id, \sigma_2'\sigma_2'\} \qquad H_5 = G.$$

Using the set of all subgroups which is $\{H_1, H_2, H_3, H_4, H_5\}$, we will find all intermediate fields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ using the correspondence between

$\mathcal{H}$ is the set of all subgroups in G, and
$\mathcal{F}$ is the set of all intermediate field of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$

defined by

$$\rho: \mathcal{H} \to \mathcal{F}$$
$$H_i \mapsto K^{H_i}$$

for all $i = 1,2,3,4$. Note that each automorphism in G defined by

$$id: \mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6} \mapsto a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6}$$

$$\sigma_2': \mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6} \mapsto a.1 - b.\sqrt{2} + c.\sqrt{3} - d.\sqrt{6}$$

$$\sigma_2': \mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6} \mapsto a.1 + b.\sqrt{2} - c.\sqrt{3} - d.\sqrt{6}$$

$$\sigma_2'\sigma_3': \mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6} \mapsto a.1 - b.\sqrt{2} - c.\sqrt{3} + d.\sqrt{6}.$$

for every $a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Therefore, the fixed of fields of each automorphism is

$$K^{\{id\}} = \{a.1 + b.\sqrt{2} + c.\sqrt{3} + d.\sqrt{6} | a, b, c, d \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$K^{\{\sigma_2'\}} = \{a.1 + c.\sqrt{3} | a, c \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{3})$$
$$K^{\{\sigma_3'\}} = \{a.1 + b.\sqrt{2} | a, b \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2})$$
$$K^{\{\sigma_2'\sigma_3'\}} = \{a.1 + d.\sqrt{6} | a, d \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{6}).$$

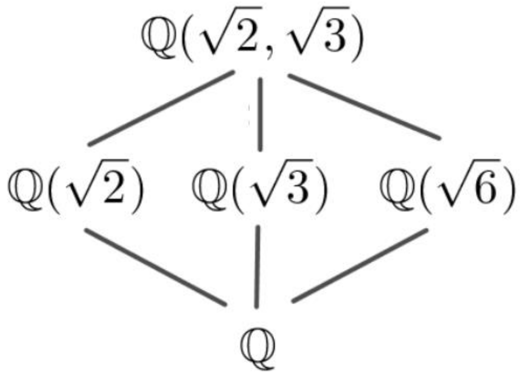
Thus, the fixed field for each subgroups are

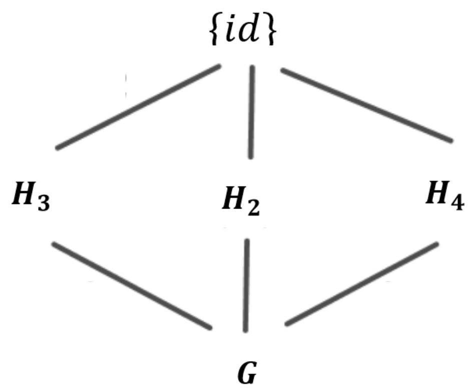
$$K^{H_1} = K^{\{id\}} = \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$K^{H_2} = K^{\{id,\sigma_2'\}} = K^{\{id\}} \cap K^{\{\sigma_2'\}} = \mathbb{Q}(\sqrt{3})$$
$$K^{H_3} = K^{\{id,\sigma_3'\}} = K^{\{id\}} \cap K^{\{\sigma_3'\}} = \mathbb{Q}(\sqrt{2})$$
$$K^{H_4} = K^{\{id,\sigma_2'\sigma_3'\}} = K^{\{id\}} \cap K^{\{\sigma_2'\sigma_3'\}} = \mathbb{Q}(\sqrt{6})$$
$$K^{H_5} = K^{G} = K^{\{id\}} \cap K^{\{\sigma_2'\}} \cap K^{\{\sigma_3'\}} \cap K^{\{\sigma_2'\sigma_3'\}} = \mathbb{Q}$$

Therefore,

$$\rho: \mathcal{H} \to \mathcal{F}$$
$$H_1 \mapsto \mathbb{Q}(\sqrt{2}, \sqrt{3})$$
$$H_2 \mapsto \mathbb{Q}(\sqrt{3})$$
$$H_3 \mapsto \mathbb{Q}(2)$$
$$H_4 \mapsto \mathbb{Q}(\sqrt{6})$$
$$H_5 \mapsto \mathbb{Q}.$$

Hence, the set of all intermediate fields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is $\{\mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{6})$ and $\mathbb{Q}$. Furthermore, we will the describe the correspondence using the diagram below





CONCLUSION

Let K/F be an extension field with its automorphism group $G = Aut(K/F)$.

1. The field K/F is Galois extension if and only if the fixed field of G is F itself.
2. If K/F is a Galois extension then there is one-one correspondence between the set of all intermediate subfields of K/F and the set of all subgroups in G.

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