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A Generalization of Chio's Condensation Method

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ABSTRACT

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Chio condensation method is a method to compute the determinant of an $n \times n$ matrix A where $a_{11} \neq 0$ by reducing the order of the matrix to an $(n-1) \times (n-1)$matrix. In this paper, we will generalize the condition where a_{11} can be equal to zero. To compute the determinant, we can choose any element of matrix A that is not equal to zero as a pivot element.

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1. Introduction

Numerous branches of mathematics, science, and engineering make regular use of determinants in various contexts. Calculating the determinants of small matrices is a simple process that can be done using the Laplace expansion by rows or columns [1]. The challenges come into play when one needs to work with very big matrices. There are just a few intriguing non-traditional ways for determining the determinant of a square matrix in the older literature.

These ways, on the other hand, are based on something called the" condensation method," which involves changing the order of the original determinant [2]. The two most popular of these methods are called Chio [3] and Dodgson's condensation[4]. In this paper, we will review the above-mentioned condensation methods and then show a new way to find the determinant of a square matrix by reducing its order one step at a time using Chio and Dodgson's determinantal identities. This will give us a determinant of order two, which is easy to find. Some researchers have worked on Chio's condensation method such as [5], [6], [7] and [8].

The Chio condensation method is a method for computing the determinant of a matrix by reducing the matrix order $n \times n$ into $n-1 \times n-1$ and suppose $a_{11} \neq 0$ as a pivot element. The Chio condensation method was first proposed by F. Chio in 1853. However, there are earlier indications of this method in C. Hermite's article published in 1849 [3]. The general form of the Chio condensation process is $\det A = \frac{\det B}{a_{11}^{n-2}}$.

The supposing that using the $\boldsymbol{a_{11}}$ element as a pivot element would be difficult if found the element $a_{11} = 0$ in a matrix. Therefore, modification of the flexible pivot of the Chiocondensation method is needed for any element that can be selected flexibly to be a pivot element. In this paper, we introduce the generalization of Chio's condensation method, when we find the a_{11} element is equal to zero. We also then make a conclusion that we can choose any element in an $n \times n$ matrix that is not equal to zero.

2. Determinant and Chio's Condensation

In this section, we will discuss the basic concept of determinants and Chio's condensation method.

2.1 Determinant

In linear algebra, the determinant is a scalar value that can be calculated for a square matrix. The determinant of matrix A is typically denoted as $\mathbf{det}\,\boldsymbol{A}$ or $|\boldsymbol{A}|$.
For a $\mathbf{2} \times \mathbf{2}$ matrix where $\mathbf{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{bmatrix}$ then the determinant of matrix A is calculated as follows:

$$\mathbf{det}(\boldsymbol{A}) = \boldsymbol{a_{11}a_{22} - a_{12}a_{21}} \tag{1}$$

If A is an $n \times n$ matrix, determinant is a scalar associated with a square matrix A and denoted as *det(A)*, or $|A|$. To determine the determinant of a $n \times n$ matrix A, a typical technique is cofactor expansion. Let $M_{i,j}$ be the minor of entery $a_{i,j}(i = 1, 2, \dots, n)$ and $j = 1, 2, \dots, n)$, which is the determinant of the sub matrix that results from deleting the i^{th} row and j^{th} column of A. If i^{th} row of A is opted for cofactor expansion then,

$$\det(A) = \sum_{j=1}^{n} a_{i,j}A_{i,j} = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j}M_{i,j} \tag{2}$$

where A_{ij} is the cofactor of entry a_{ij} such that $A_{1j} = (-1)^{1+j}M_{1j}$. Similarly, the cofactor expansion along the j^{th} column would be

$$\det(A) = \sum_{j=1}^{n} a_{1j}A_{1j} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}M_{1j} \tag{3}$$

Another common method used to compute the determinant of a large matrix is elementary row operation. The next theorem shows how an elementary row operation computes the determinant of the $n \times n$ matrix.

2.2 Chio Condensation Method

In this subsection, we begin with a statement of the Chio Condensation Method theorem:
Theorem 1. [2] *Let A be an $n \times n$ matrix and suppose $a_{11} \neq 0$. Let B denoted the $(n-1) \times (n-1)$ matrix obtained by replacing each element a_{1j} by $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$. Then $|A| = \frac{|B|}{a_{11}^{n-2}}$, were*

$$|\mathrm{B}| = \begin{vmatrix} a_{11} & a_{1(j+i)} \\ a_{(i+1)1} & a_{(i+1)(j+1)} \end{vmatrix} \tag{4}$$

Proof. Let A be an $n \times n$ matrix, denoted by

$$\mathrm{A}_{\mathrm{n\times n}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{bmatrix}$$

then we can compute the determinant of matrix A using **Eq. (3)** as follow,

$$|\mathrm{A}_{\mathrm{n\times n}}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix} \tag{5}$$

Multiply each row of **Eq. (5)** by a_{11} except the first row and then from Theorem ?? we have

$$|\mathrm{A}_{\mathrm{n\times n}}| = \frac{1}{a_{11}^{n-1}} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21}a_{11} & a_{22}a_{11} & a_{23}a_{11} & a_{24}a_{11} & \cdots & a_{2n}a_{11} \\ a_{31}a_{11} & a_{32}a_{11} & a_{33}a_{11} & a_{34}a_{11} & \cdots & a_{3n}a_{11} \\ a_{41}a_{11} & a_{42}a_{11} & a_{43}a_{11} & a_{44}a_{11} & \cdots & a_{4n}a_{11} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} & a_{n2}a_{11} & a_{n3}a_{11} & a_{n4}a_{11} & \cdots & a_{nn}a_{11} \end{vmatrix} \tag{6}$$

Multiply both side by a_{11}^{n-1} from **Eq. (6)** then we get the following result.

$$a_{11}^{n-1}|\mathrm{A}_{\mathrm{n\times n}}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21}a_{11} & a_{22}a_{11} & a_{23}a_{11} & a_{24}a_{11} & \cdots & a_{2n}a_{11} \\ a_{31}a_{11} & a_{32}a_{11} & a_{33}a_{11} & a_{34}a_{11} & \cdots & a_{3n}a_{11} \\ a_{41}a_{11} & a_{42}a_{11} & a_{43}a_{11} & a_{44}a_{11} & \cdots & a_{4n}a_{11} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} & a_{n2}a_{11} & a_{n3}a_{11} & a_{n4}a_{11} & \cdots & a_{nn}a_{11} \end{vmatrix} \tag{7}$$

Then we do the elementary row operations. Firstly, subtract second row from **Eq. (7)** by the multiplication of a_{21} with the first row.

$$a_{11}^{n-1}|\mathrm{A}_{\mathrm{n\times n}}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22}a_{11} - a_{12}a_{21} & a_{23}a_{11} - a_{13}a_{21} & a_{24}a_{11} - a_{14}a_{21} & \cdots & a_{2n}a_{11} - a_{1n}a_{21} \\ a_{31}a_{11} & a_{32}a_{11} & a_{33}a_{11} & a_{34}a_{11} & \cdots & a_{3n}a_{11} \\ a_{41}a_{11} & a_{42}a_{11} & a_{43}a_{11} & a_{44}a_{11} & \cdots & a_{4n}a_{11} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} & a_{n2}a_{11} & a_{n3}a_{11} & a_{n4}a_{11} & \cdots & a_{nn}a_{11} \end{vmatrix}$$

after that, subtract second row until the n^{th} row from **Eq. (7)** by the multiplication of $a_{21}, a_{31}, a_{41}, \cdots, a_{n1}$ with the first row.

$$a_{11}^{n-1}|\mathrm{A}_{\mathrm{n\times n}}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22}a_{11} - a_{12}a_{21} & a_{23}a_{11} - a_{13}a_{21} & a_{24}a_{11} - a_{14}a_{21} & \cdots & a_{2n}a_{11} - a_{1n}a_{21} \\ 0 & a_{32}a_{11} - a_{12}a_{31} & a_{33}a_{11} - a_{13}a_{31} & a_{34}a_{11} - a_{14}a_{31} & \cdots & a_{3n}a_{11} - a_{1n}a_{31} \\ a_{41}a_{11} & a_{42}a_{11} & a_{43}a_{11} & a_{44}a_{11} & \cdots & a_{4n}a_{11} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} & a_{n2}a_{11} & a_{n3}a_{11} & a_{n4}a_{11} & \cdots & a_{nn}a_{11} \end{vmatrix}$$

$$a_{11}^{n-1}|\mathrm{A}_{\mathrm{n\times n}}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22}a_{11}-a_{12}a_{21} & a_{23}a_{11}-a_{13}a_{21} & a_{24}a_{11}-a_{14}a_{21} & \cdots & a_{2n}a_{11}-a_{1n}a_{21} \\ 0 & a_{32}a_{11}-a_{12}a_{31} & a_{33}a_{11}-a_{13}a_{31} & a_{34}a_{11}-a_{14}a_{31} & \cdots & a_{3n}a_{11}-a_{1n}a_{31} \\ 0 & a_{42}a_{11}-a_{12}a_{41} & a_{43}a_{11}-a_{13}a_{41} & a_{44}a_{11}-a_{14}a_{41} & \cdots & a_{4n}a_{11}-a_{1n}a_{41} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} & a_{n2}a_{11} & a_{n3}a_{11} & a_{n4}a_{11} & \cdots & a_{nn}a_{11} \end{vmatrix}$$

$$a_{11}^{n-1}|\mathrm{A}_{\mathrm{n\times n}}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22}a_{11}-a_{12}a_{21} & a_{23}a_{11}-a_{13}a_{21} & a_{24}a_{11}-a_{14}a_{21} & \cdots & a_{2n}a_{11}-a_{1n}a_{21} \\ 0 & a_{32}a_{11}-a_{12}a_{31} & a_{33}a_{11}-a_{13}a_{31} & a_{34}a_{11}-a_{14}a_{31} & \cdots & a_{3n}a_{11}-a_{1n}a_{31} \\ 0 & a_{42}a_{11}-a_{12}a_{41} & a_{43}a_{11}-a_{13}a_{41} & a_{44}a_{11}-a_{14}a_{41} & \cdots & a_{4n}a_{11}-a_{1n}a_{41} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}a_{11}-a_{12}a_{n1} & a_{n3}a_{11}-a_{13}a_{n1} & a_{n4}a_{11}-a_{14}a_{n1} & \cdots & a_{nn}a_{11}-a_{1n}a_{n1} \end{vmatrix}$$

$$a_{11}^{n-1}|\mathrm{A}_{\mathrm{n\times n}}| = \begin{vmatrix} a_{22}a_{11}-a_{12}a_{21} & a_{23}a_{11}-a_{13}a_{21} & a_{24}a_{11}-a_{14}a_{21} & \cdots & a_{2n}a_{11}-a_{1n}a_{21} \\ a_{32}a_{11}-a_{12}a_{31} & a_{33}a_{11}-a_{13}a_{31} & a_{34}a_{11}-a_{14}a_{31} & \cdots & a_{3n}a_{11}-a_{1n}a_{31} \\ a_{42}a_{11}-a_{12}a_{41} & a_{43}a_{11}-a_{13}a_{41} & a_{44}a_{11}-a_{14}a_{41} & \cdots & a_{4n}a_{11}-a_{1n}a_{41} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n2}a_{11}-a_{12}a_{n1} & a_{n3}a_{11}-a_{13}a_{n1} & a_{n4}a_{11}-a_{14}a_{n1} & \cdots & a_{nn}a_{11}-a_{1n}a_{n1} \end{vmatrix}$$

Multiply both side by $\frac{1}{a_{11}^{n-1}}$ then we have

$$|\mathrm{A}_{\mathrm{n\times n}}| = \frac{1}{a_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{41} & a_{4n} \end{vmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{n1} & a_{n4} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix} \tag{8}$$

Eq. (8) has the following form.

$$|\mathrm{A}_{\mathrm{n\times n}}| = \frac{1}{a_{11}^{n-2}}|B|$$

□

Using **Theorem 1** we can compute the determinant of matrices easily. Therefore, we construct the algorithm for computing the determinant of an $n \times n$ matrix based on **Theorem 1** as follows:

Algorithm 1: **Theorem 1** Condensation Method

Input: $A_{n\times n}$ where $a_{11} \neq 0$

Output: the determinant of matrix A

We do the Chio's condensation method in the following steps:

1. Choose $a_{11} \neq 0$ as a pivot element
2. transform matrix A by reducing the dimension $(n-1) \times (n-1)$ matrix B as in Eq. (4)
3. repeat the step 2 by reducing of matrix B is equal to 2×2.
4. Calculate $|A_{n\times n}| = \frac{1}{a_{11}^{n-2}}|B|$

Example 1. Consider an 4×4 matrix A as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 1 & 8 & 3 & 7 \\ 3 & 6 & 4 & 5 \end{bmatrix}$$

then we can compute *det(A)* using **Theorem 1**. First of All, we choose $a_{11} = 1$ as a pivot element. Then we construct an $(4-1) \times (4-1)$ matrix B as defined in ***Algorithm 1***.

$$B = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 8 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 7 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -9 & -18 & -27 \\ 6 & -1 & 3 \\ 0 & -5 & -7 \end{bmatrix}$$

repeating the process until we have a 2×2 matrix B as follows.

$$B = \begin{bmatrix} 117 & 135 \\ 45 & 63 \end{bmatrix}$$

then we have

$$|A| = \frac{1}{9^{3-2}} \begin{vmatrix} 117 & 135 \\ 45 & 63 \end{vmatrix} = \frac{1}{-9}(1296) = -144$$

3. Generalisation of Chio Method

In this section, we modify the flexible pivot of Chio's condensation method where $a_{11} = 0$.

Theorem 2 (*Chio's condensation* $a_{11} = 0$). *Let A be an* $n \times n$ *matrix where* $a_{11} = 0$. *Let any element of matrix A, i.e.* $a_{r,s}$ *as a pivot element with r-th and s-th column. Let B be an* $(n-1) \times (n-1)$ *defined by*

$$B = (b_{ij}) = \begin{cases} \begin{vmatrix} a_{ij} & a_{is} \\ a_{rj} & a_{rs} \end{vmatrix} & \text{if } i < r \text{ and } j < s \\ -\begin{vmatrix} a_{is} & a_{i(j+i)} \\ a_{rs} & a_{r(j+1)} \end{vmatrix} & \text{if } i \le s \text{ and } s \le j \le r \\ -\begin{vmatrix} a_{ij} & a_{rs} \\ a_{(i+1)j} & a_{(i+1)s} \end{vmatrix} & \text{if } i = r \text{ and } s \le j \le r \\ \begin{vmatrix} a_{rs} & a_{i(j+1)} \\ a_{(i+1)s} & a_{i+1}a_{j+1} \end{vmatrix} & \text{if } i = r \text{ and } j < s \end{cases} \tag{9}$$

for $i, j \in |n-1|$. *Then* $|A| = \frac{(-1)^{r+x}}{a_{rx}^{n-2}} |B|$.

Proof. Let A be an $n \times n$ matrix denoted by

$$A_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rs} & \cdots & a_{rn} \\ \cdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \quad a_{11} = 0 \tag{10}$$

Then we can compute determinant of matrix A in **Eq. (10)** using **Eq. (3)** as follows,

$$|A_{n\times n}| = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rs} & \cdots & a_{rn} \\ \cdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \quad a_{11} = 0 \tag{11}$$

Element a_{rs} in matrix A as a pivot, with $a_{rs} \neq a_{11}$ and $a_{rs} \neq 0$ Multiply each row of **Eq. (12)** by a_{rs} except the r^{th} row and s^{th} column and then from Theorem ?? we have,

$$|A_{n\times n}| = \frac{1}{a_{rs}^{n-1}} \begin{vmatrix} a_{rs}a_{11} & \cdots & a_{rs}a_{1j} & \cdots & a_{rs}a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{rs}a_{r1} & \cdots & a_{rs}a_{rs} & \cdots & a_{rs}a_{rn} \\ \cdots & \ddots & \vdots & \ddots & \vdots \\ a_{rs}a_{n1} & \cdots & a_{rs}a_{nj} & \cdots & a_{rs}a_{nn} \end{vmatrix}$$

Multiply by both side by a_{rs}^{n-1} then we get the following result:

$$a_{rs}^{n-1}|A_{n\times n}| = \begin{vmatrix} a_{rs}a_{11} & \cdots & a_{rs}a_{1j} & \cdots & a_{rs}a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rs} & \cdots & a_{rn} \\ \cdots & \ddots & \vdots & \ddots & \vdots \\ a_{rs}a_{n1} & \cdots & a_{rs}a_{nj} & \cdots & a_{rs}a_{nn} \end{vmatrix}$$

Then we do the elementary row operations to make each element on the s^{th} column except the r^{th}row gets to 0.

$$a_{rs}^{n-1}|A_{n\times n}| = \begin{vmatrix} a_{rs}a_{11} - a_{r1}a_{is} & \cdots & 0 & \cdots & a_{rs}a_{1n} - a_{rn}a_{is} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rs} & \cdots & a_{rn} \\ \cdots & \ddots & \vdots & \ddots & \vdots \\ a_{rs}a_{n1} - a_{r1}a_{is} & \cdots & 0 & \cdots & a_{rs}a_{nn}a_{rn}a_{ns} \end{vmatrix}$$

Using **Theorem 2**

$$a_{rs}^{n-1}|A_{n\times n}| = a^{rs} \begin{vmatrix} a_{rs}a_{11} - a_{r1}a_{is} & \cdots & -(a_{rn}a_{is} - a_{rs}a_{1n}) \\ \vdots & \vdots & \vdots \\ -(a_{r1}a_{is} - a_{rs}a_{n1}) & \cdots & a_{rs}a_{nn}a_{rn}a_{ns} \end{vmatrix} \tag{12}$$

Multiply both side by $\frac{1}{a_{rs}^{n-2}}$ fom **Eq. (12)** obtained,

$$|A_{n\times n}| = \frac{1}{a_{rs}^{n-2}} \begin{vmatrix} a_{rs}a_{11} - a_{r1}a_{is} & \cdots & -(a_{rn}a_{is} - a_{rs}a_{1n}) \\ \vdots & \vdots & \vdots \\ -(a_{r1}a_{is} - a_{rs}a_{n1}) & \cdots & a_{rs}a_{nn}a_{rn}a_{ns} \end{vmatrix} \tag{13}$$

Eq. (13) can be expressed as,

$$|A_{n\times n}| = \frac{1}{a_{rs}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{1s} \\ a_{r1} & a_{rs} \end{vmatrix} & \cdots & -\begin{vmatrix} a_{1s} & a_{1n} \\ a_{rs} & a_{rn} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ -\begin{vmatrix} a_{r1} & a_{rs} \\ a_{n1} & a_{ns} \end{vmatrix} & \cdots & \begin{vmatrix} a_{rs} & a_{rn} \\ a_{ns} & a_{nn} \end{vmatrix} \end{vmatrix}$$

$$a_{rs}^{n-1}|A_{n\times n}| = \begin{cases} \begin{vmatrix} a_{ij} & a_{is} \\ a_{rj} & a_{rs} \end{vmatrix} & if\ i < r\ and\ j < s \\ -\begin{vmatrix} a_{is} & a_{i(j+i)} \\ a_{rs} & a_{r(j+1)} \end{vmatrix} & if\ i \le s\ and\ s \le j \le r \\ -\begin{vmatrix} a_{ij} & a_{rs} \\ a_{(i+1)j} & a_{(i+1)s} \end{vmatrix} & if\ i = r\ and\ s \le j \le r \\ \begin{vmatrix} a_{rs} & a_{i(j+1)} \\ a_{(i+1)s} & a_{i+1}a_{j+1} \end{vmatrix} & if\ i = r\ and\ j < s \end{cases}$$

$$|A| = \frac{(-1)^{r+s}}{a_{rs}^{n-2}}\,|B|$$

□

Algorithm 2: **Theorem 2** Generalization of Chio Method

Input: $A_{n\times n}$ where $a_{11} = 0$

Output: the determinant of matrix A

We do Chio's condensation method in the following steps:

1. Choose $a_{r,s} \ne 0$ and $a_{rs} \ne a_{11}$ as a pivot element
2. transform matrix A by reducing the dimension $(n-1) \times (n-1)$ matrix B as in **Eq. (9)**
3. repeat the step 2 by reducing of matrix B is equal to 2×2.
4. Calculate $|A_{n\times n}| = \frac{(-1)^{r+s}}{a_{rs}^{n-2}}\,|B|$

Here we give an example.

Example 2. *Let A be a* 4×4 *matrix as follows*

$$A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 3 & -2 & 8 & 5 \\ 2 & 1 & 3 & 1 \\ 4 & 5 & 4 & -3 \end{bmatrix}$$

then we can compute the determinant of matrix A using Algorithm 2. First, we choose $a_{32} = 1$ *as a pivot element then we construct matrix B by reducing the order of the matrix as follows.*

$$B = \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 8 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 5 & -3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -4 & -3 & -1 \\ 7 & 14 & 7 \\ -6 & -11 & -8 \end{bmatrix}$$

Then we compute the determinant of matrix A as below

$$|A| = \frac{(-1)^{3+2}}{1^{4-2}} \begin{vmatrix} -4 & -3 & -1 \\ 7 & 14 & 7 \\ -6 & -11 & -8 \end{vmatrix}$$

Since the order of matrix B is still 3×3 *then we construct matrix* B_{new} *by reducing the order of matrix B and we choose* $b_{23} = 7$ *as a pivot element. Then we have*

$$B_{new} = \begin{bmatrix} \begin{vmatrix} -4 & -1 \\ 7 & 7 \end{vmatrix} & \begin{vmatrix} -3 & -1 \\ 14 & 7 \end{vmatrix} \\ \begin{vmatrix} 7 & 7 \\ -6 & -8 \end{vmatrix} & \begin{vmatrix} 14 & 7 \\ -11 & -8 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -21 & -7 \\ 14 & 35 \end{bmatrix}$$

we repeat the process to compute the determinant of A as follows.

$$|A| = (-1)\frac{(-1)^{2+3}}{7^{3-2}} \begin{vmatrix} -21 & -7 \\ 14 & 35 \end{vmatrix} = \frac{1}{7}(-637) = -91$$

We then generalize that the pivot element does not depend on $a_{11} = 0$ or $a_{11} \neq 0$ as in the following Remark.

Remark 1. *Let* $A = (a_{ij})$ *be an* $n \times n$ *matrix for* $i, j \in |n|$ *then we can choose* $a_{rs} \neq 0$ *as a pivot element. Then we can compute* $|A| = \frac{(-1)^{r+s}}{a_{rs}^{n-2}} |B|$.

4. Conclusions

In this paper presented the generalization of Chio's condensation method for computing the determinant of $n \times n$ matrices where $a_{11} = 0$. Let A be an $n \times n$ matrix, the pivot can be selected from any element a_{rs} on the r^{th} row and s^{th} column and we can build an $(n-1) \times (n-1)$ matrix B. The determinant of matrix A can be defined by

$$|A| = \frac{(-1)^{r+s}}{a_{rs}^{n-2}} |B|$$

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