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On Local-Strong Rainbow Connection Numbers On Generalized Prism Graphs And Generalized Antiprism Graphs

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ABSTRACT

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Rainbow geodesic is the shortest path that connects two different vertices in graph $\boldsymbol{G}$ such that every edge of the path has different colors. The strong rainbow connection number of a graph $\boldsymbol{G}$, denoted by src(G), is the smallest number of colors required to color the edges of G such that there is a rainbow geodesic for each pair of vertices. The $\boldsymbol{d}$-local strong rainbow connection number, denoted by $\boldsymbol{lsrc_d}$, is the smallest number of colors required to color the edges of $\boldsymbol{G}$ such that any pair of vertices with a maximum distance $\boldsymbol{d}$ is connected by a rainbow geodesic. This paper contains some results of $\boldsymbol{lsrc_d}$ of generalized prism graphs $(\boldsymbol{P_m} \times \boldsymbol{C_n})$ and generalized antiprism graphs $(\boldsymbol{A_m^n})$ for values of $\boldsymbol{d} = \boldsymbol{2}, \boldsymbol{d} = \boldsymbol{3}$, and $\boldsymbol{d} = \boldsymbol{4}$.

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1. Introduction

Chartrand, Johns, McKoen, and Zhang [1] introduced strong rainbow connection numbers. Let c be edge coloring of a connected graph G. For any two edges u and v in G, a rainbow geodesic is the shortest rainbow path $u - v$. G is a strong rainbow connected graph if any two edges of G are connected by a rainbow geodesic. An edge coloring c that has rainbow geodesic is called strong rainbow coloring of G. A strong rainbow connection number of connected G, denoted by $src(G)$, is defined as the smallest number of colors in a strong rainbow coloring of G [2]. A rainbow connection can be applied to the secure transfer of classified information [3]. There are several generalizations of rainbow connection. For instance, rainbow k-connectivity [4], k-rainbow index [5], the vertex version [6], total version [7], directed version [8], and rainbow connection for hypergraphs [9]. The reader can check the more information on [2] and [10].

Septyanto & Sugeng considered to localize some properties of the strong rainbow connection. Instead of considering every pair of vertices connected by a rainbow path, they considered only pair of vertices that has distance up to d, for a positive integer d. The d-local strong rainbow coloring is an edge coloring so that any two vertices with a maximum distance d are connected by a rainbow geodesic. The d-local strong rainbow connection number of a connected graph G, $lsrc_d(G)$, is the smallest number of colors which is needed in the d-local strong rainbow coloring of this graph [11]. Furthermore, Septyanto & Sugeng determined the value of $lsrc_d$ for cycle graphs (C_n). Nugroho & Sugeng [12] explored and found the value of $lsrc_d$ for a prism graph. The prism graph can be defined as the Cartesian product $P_2 \times C_n$, where P_2 is a path with two vertices and C_n is a cycle with n vertices. Thus, a prism graph has order $2n$ and size $3n$ [13].

The previous result raise a question on what is the value of $\boldsymbol{lsrc_d}$ would be if the prism graph is generalized to $P_m \times C_n$. The generalized prism graph is the Cartesian product between path P_m and cycle C_n, that is $P_m \times C_n$ [14]. The research is then extended to generalized antiprims graph which constructed by adding an edge diagonally for each cycle subgraph C_4 in the graph [15].

2. Known Result

There are several studies that discuss local strong rainbow coloring and its relation to $\boldsymbol{lsrc_d}$. In a previous study, Septyanto and Sugeng [2] determined the $\boldsymbol{d}$-local strong rainbow connection number for circle graphs. Darmawan and Dafik [16] determined the value of $\boldsymbol{rc}$ and $\boldsymbol{src}$ of generalized prism.

Theorem 1 [2]

For any graph G and a positive integer d,

$$lrc_d(G) \le lrc_{d+1}(G) \text{ and } lsrc_d(G) \le lsrc_{d+1}(G).$$

Theorem 2 [11]

If $n \ge 3$ and $d \le n/2$, then $lsrc_d(C_n) = \left\lceil \frac{n}{\lfloor n/d \rfloor} \right\rceil$.

Theorem 3 [12]

If $m \ge 3$, then $lsrc_d(P_m) = d$.

Theorem 4 [12]

For $n \ge 3$, $d \le \frac{n}{2}$, $lsrc_d(P_2 \times C_n) = \left\lceil \frac{n}{\lfloor n/d \rfloor} \right\rceil$.

Theorem 5 [16]

Let $G = P_m \times C_n$ be prism graphs, for $m \ge 3$ and $n \ge 1$, $rc(G) = src(G) = \begin{cases} m; & for\ n = 3, \\ \left\lceil \frac{n}{2} \right\rceil + (m-1); & for\ n \ge 4 \end{cases}$.

3. Results And Discussion

It will be shown that there exists a rainbow coloring such that every pair of edges of maximum distance d is connected by a rainbow geodesic. In the following theorems, we determined the value of $lsrc_d$ for generalized prism graphs and generalized antiprism graph, for $d = 2$, $d = 3$ and $d = 4$.

Theorem 6

For $n \geq 4$, $lsrc_2(P_m \times C_n) = \begin{cases} 2; n\ even, \\ 3; n\ odd. \end{cases}$

Proof.

Let $G = P_m \times C_n$, where $V(G) = \left\{v_i^{(j)} \middle| i = 1,2,3, \dots, n; j = 1,2,3, \dots, m\right\}$ and

$E(G) = \left\{v_i^{(j)} v_{i+1}^{(j)} \middle| i = 1,2,3, \dots, n; j = 1,2,3, \dots, m\right\} \cup \left\{v_i^{(j)} v_i^{(j+1)} \middle| i = 1,2,3, \dots, n; j = 1,2,3, \dots, m-1\right\}$.

Consider 2 cases based on the values of n.

Case 1: n even

To prove that $lsrc_2(P_m \times C_n) = 2$, we define an edge coloring $c_1 : E(G) \to \{1,2\}$ as follows:

i. $c_1\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 1, i\text{ even}, \\ 2, i\ odd, \end{cases}$

$i = 1,2,3, \dots, n-1$ and $j = 1,2,3, \dots, m$.

ii. $c_1\left(v_i^{(j)} v_1^{(j)}\right) = 2,\ i = n$ and $j = 1,2,3, \dots, m$.

iii. $c_1\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, i\text{ odd};\ j\text{ odd}, \\ 2, i\text{ odd};\ j\text{ even}. \end{cases}$

iv. $c_1\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 2, i\text{ even};\ j\text{ odd}, \\ 1, i\text{ even}; j\text{ odd}. \end{cases}$

Case 2: n odd

To prove that $lsrc_2(P_m \times C_n) = 3$, we define an edge coloring $c_2 : E(G) \to \{1,2,3\}$ as follows:

i. $c_2\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 1, i\text{ even}, \\ 2, i\text{ odd}, \end{cases}$

$i = 1,2,3, \dots, n-1$ and $j = 1,2,3, \dots, m$.

ii. $c_2\left(v_n^{(j)} v_1^{(j)}\right) = 3, j = 1,2,3, \dots, m$.

iii. $c_2\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, i\text{ odd};\ j\text{ odd}, \\ 2, i\text{ odd};\ j\text{ even}. \end{cases}$

iv. $c_2\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 2, \quad i\text{ even};\ j\text{ odd}, \\ 3, i\text{ even};\ j\text{ even}, \end{cases}$

v. $c_2\left(v_n^{(j)} v_i^{(j+1)}\right) = \begin{cases} 3,\ j\text{ odd}, \\ 1,\ j\text{ even}, \end{cases} \quad j = 1,2,3, \dots, m.$

By looking of all possibilities of every geodesic path should be rainbow for distance two we cannot have less colors, then we can conclude that for $n \geq 4$, $lsrc_2(P_m \times C_n) = \begin{cases} 2; n\text{ even}, \\ 3; n\text{ odd}. \end{cases}$ ■

Figure 1. shows the example of the local strong rainbow coloring of generalized prism for $d = 2$.

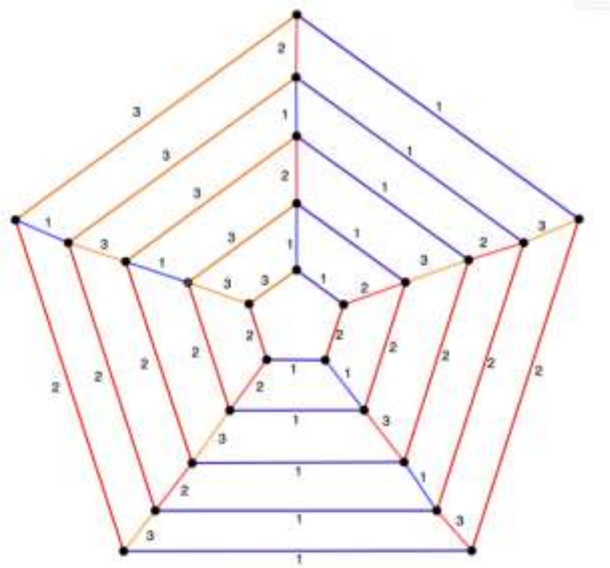

Figure 1. Example of 2-local strong rainbow coloring on prism graph $P_5 \times C_5$

Theorem 7

$$For\ n \geq 6,\ lsrc_3(P_m \times C_n) = \begin{cases} 3; m = 3,4; 3|n \\ 4; m > 4; 3|n \\ 4; 3 \nmid n \end{cases}$$

Proof.

Case 1: $3|n$

For $n = 6, 9, 12, \ldots$ it will be shown that $lsrc_3(P_m \times C_n) = 3$. Define the edge coloring $c_3 : E(G) \to \{1,2,3,4\}$ as follows:

i. $c_3\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 1, & i = 1 \bmod 3, \\ 2, & i = 2 \bmod 3, \\ 3, & i = 0 \bmod 3, \end{cases}$

$i = 1, 2, 3, 4, 5, 6, \ldots, n-1$ and $j = 1,2,3, \ldots, m$.

ii. $c_3\left(v_n^{(j)} v_1^{(j)}\right) = 3,\ j = 1,2,3, \ldots, m$.

iii. $c_3\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 3, & j = 1 \bmod 4, \text{and } i = 1 \bmod 3, \\ 2, & j = 2 \bmod 4, \text{and } i = 1 \bmod 3, \\ 1, & j = 3 \bmod 4, \text{and } i = 1 \bmod 3, \\ 4, & j = 0 \bmod 4, \text{and } i = 1 \bmod 3. \end{cases}$

iv. $c_3\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, & j = 1 \bmod 4, \text{and } i = 2 \bmod 3, \\ 3, & j = 2 \bmod 4, \text{and } i = 2 \bmod 3, \\ 2, & j = 3 \bmod 4, \text{and } i = 2 \bmod 3, \\ 4, & j = 4 \bmod 4, \text{and } i = 2 \bmod 3. \end{cases}$

v. $c_3\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 2, & j = 1 \bmod 4, \text{and } i = 0 \bmod 3, \\ 1, & j = 2 \bmod 4, \text{and } i = 0 \bmod 3, \\ 3, & j = 3 \bmod 4, \text{and } i = 0 \bmod 3, \\ 4, & j = 4 \bmod 4, \text{and } i = 0 \bmod 3. \end{cases}$

Case 2: $3 \nmid n$

The edge coloring of $P_m \times C_n$ in general for $3 \nmid n$ can be shown using $n = 7$. Define the edge coloring $c_4 : E(G) \to \{1,2,3,4\}$ as follows:

i. $c_4\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 1, & i = 1, 4, \\ 2, & i = 2,5, \\ 3, & i = 3,6, \end{cases}$

$j = 1,2,3, \ldots, m$.

ii. $c_4\left(v_i^{(j)} v_1^{(j)}\right) = 4,\ i = 7$ and $j = 1,2,3, \ldots, m$.

iii. $c_4\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, & i = 1,4; j = 1, \\ 2, & i = 1,4; j = 2 \bmod 3, \\ 3, & i = 1,4; j = 0 \bmod 3, \\ 4, & i = 1,4; j = 1 \bmod 3. \end{cases}$

iv. $c_4\left(v_2^{(j)} v_2^{(j+1)}\right) = \begin{cases} 2, & j = 1, \\ 3, & j = 2 \bmod 3, \\ 1, & j = 0 \bmod 3, \\ 4, & j = 1 \bmod 3. \end{cases}$

v. $c_4\left(v_3^{(j)} v_3^{(j+1)}\right) = \begin{cases} 3, & j = 1, \\ 1, & j = 2 \bmod 3, \\ 2, & j = 0 \bmod 3, \\ 4, & j = 1 \bmod 3. \end{cases}$

vi. $c_4\left(v_5^{(j)} v_5^{(j+1)}\right) = \begin{cases} 2, & j = 1, \\ 3, & j = 2 \bmod 3, \\ 4, & j = 0 \bmod 3, \\ 1, & j = 1 \bmod 3. \end{cases}$

vii. $c_4\left(v_6^{(j)} v_6^{(j+1)}\right) = \begin{cases} 3, & j = 1, \\ 4, & j = 2 \bmod 3, \\ 1, & j = 0 \bmod 3, \\ 2, & j = 1 \bmod 3. \end{cases}$

viii. $c_4\left(v_7^{(j)} v_7^{(j+1)}\right) = \begin{cases} 4, & j = 1, \\ 1, & j = 2 \bmod 3, \\ 2, & j = 0 \bmod 3, \\ 3, & j = 1 \bmod 3. \end{cases}$

By looking of all possibilities of every geodesic path should be rainbow for distance two we cannot have less colors, then we can conclude that for $n \geq 6$, $lsrc_3(P_m \times C_n) = \begin{cases} 3; m = 3,4; 3|n, \\ 4; m > 4; 3|n, \\ 4; 3 \nmid n. \end{cases}$ ■

Figure 2 shows the example of the local strong rainbow coloring of generalized prism for $d = 3$.

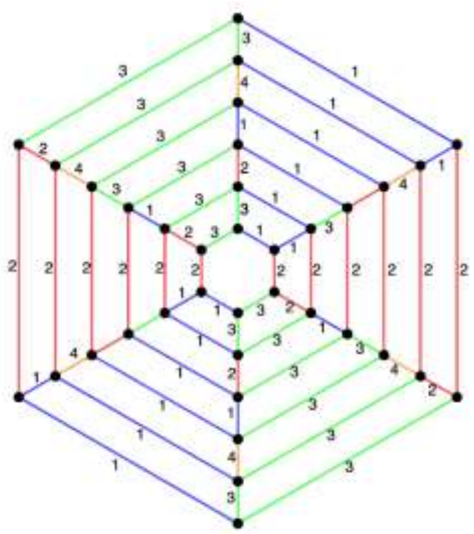

Figure 2. Example of 3-local strong rainbow coloring on prism graph $P_6 \times C_6$

Theorem 8

For $n \geq 8$, $lsrc_4(P_m \times C_n) = \begin{cases} 4; m = 3,4; 4|n, \\ 5; m > 4; 4|n, \\ 5; \ 4 \nmid n. \end{cases}$

Proof:

Consider the following cases.

Case 1: $n = 8, m \leq 4$

For the case which $m = 3$ dan $m = 4$, it will be shown that $lrsc_4(P_4 \times C_8) = 4$. Define the edge coloring $c_5: E(G) \to \{1,2,3,4\}$ as follows:

i. $c_5\left(v_i^{(j)}v_{i+1}^{(j)}\right) = \begin{cases} 1, i = 1, 5, \\ 2, i = 2,6, \\ 3, i = 3,7, \\ 4, i = 4, \end{cases}$

$j = 1, 2, 3,4.$

ii. $c_5\left(v_8^{(j)}v_1^{(j)}\right) = 4, j = 1,2,3,4.$

iii. $c_5\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 2, i = 1,5; j = 1, \\ 3, i = 1,5; j = 2, \\ 4, i = 1,5; j = 3. \end{cases}$

iv. $c_5\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 3, i = 2,6; j = 1, \\ 4, i = 2,6; j = 2, \\ 1, i = 2,6; j = 3. \end{cases}$

v. $c_5\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 4, i = 3,7; j = 1, \\ 1, i = 3,7; j = 2, \\ 2, i = 3,7; j = 3. \end{cases}$

vi. $c_5\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 1, i = 4,8; j = 1, \\ 2, i = 4,8; j = 2, \\ 3, i = 4,8; j = 3. \end{cases}$

Case 2: $n = 8, m > 4$

To show that $lrsc_4(P_5 \times C_8) = 5$, define the edge coloring $c_6: E(G) \to \{1,2,3,4,5\}$, as follows:

i. $c_6\left(v_i^{(j)}v_{i+1}^{(j)}\right) = \begin{cases} 1, \ i = 1, 5, \\ 2, \ i = 2,6, \\ 3, \ i = 3,7, \\ 4, \ i = 4. \end{cases}$

$j = 1, 2, 3,4,5$

ii. $c_6\left(v_8^{(j)}v_1^{(j)}\right) = 4, \ j = 1, 2, 3,4,5.$

iii. $c_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 2, i = 1,5; j = 1, \\ 3, i = 1,5; j = 2, \\ 4, i = 1,5; j = 3, \\ 5, i = 1,5; j = 4. \end{cases}$

iv. $c_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 3, i = 2,6; j = 1, \\ 4, i = 2,6; j = 2, \\ 1, i = 2,6; j = 3, \\ 5, i = 2,6; j = 4. \end{cases}$

v. $c_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 4, i = 3,7; j = 1, \\ 1, i = 3,7; j = 2, \\ 2, i = 3,7; j = 3, \\ 5, i = 3,7; j = 4. \end{cases}$

vi. $c_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 1, i = 4,8; j = 1, \\ 2, i = 4,8; j = 2, \\ 3, i = 4,8; j = 3, \\ 5, i = 4,8; j = 4. \end{cases}$

Case 3: $4 \nmid n$

In this case, we use $n = 9$ as it is the smallest value in which the concept applies. To show that $lrsc_4(P_m \times C_8) = 5$, we define the edge coloring $c_7: E(G) \to \{1,2,3,4,5\}$, as follows:

i. $c_7\left(v_i^{(j)}v_{i+1}^{(j)}\right) = \begin{cases} 1, \ i = 1, 5, \\ 2, \ i = 2,6, \\ 3, \ i = 3,7, \\ 4, \ i = 4,8. \end{cases}$

$j = 1, 2, 3, \dots, m.$

ii. $c_7\left(v_9^{(j)}v_1^{(j)}\right) = 5, j = 1,2,3, \dots, m.$

iii. $c_7\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 2, i = 1{,}5; j = 1 \bmod 4, \\ 3, i = 1{,}5; j = 2 \bmod 4, \\ 4, i = 1{,}5; j = 3 \bmod 4, \\ 5, i = 1{,}5; j = 0 \bmod 4. \end{cases}$

iv. $c_7\left(v_2^{(j)}v_2^{(j+1)}\right) = \begin{cases} 3, j = 1 \bmod 4, \\ 4, j = 2 \bmod 4, \\ 1, j = 3 \bmod 4, \\ 5, j = 0 \bmod 4. \end{cases}$

v. $c_7\left(v_3^{(j)}v_3^{(j+1)}\right) = \begin{cases} 4, j = 1 \bmod 4, \\ 1, j = 2 \bmod 4, \\ 2, j = 3 \bmod 4, \\ 5, j = 0 \bmod 4. \end{cases}$

vi. $c_7\left(v_4^{(j)}v_4^{(j+1)}\right) = \begin{cases} 1, j = 1 \bmod 4, \\ 2, j = 2 \bmod 4, \\ 3, j = 3 \bmod 4, \\ 5, j = 0 \bmod 4. \end{cases}$

vii. $c_7\left(v_6^{(j)}v_6^{(j+1)}\right) = \begin{cases} 3, j = 1 \bmod 4, \\ 4, j = 2 \bmod 4, \\ 5, j = 3 \bmod 4, \\ 1, j = 0 \bmod 4. \end{cases}$

viii. $c_7\left(v_7^{(j)}v_7^{(j+1)}\right) = \begin{cases} 4, j = 1 \bmod 4, \\ 5, j = 2 \bmod 4, \\ 1, j = 3 \bmod 4, \\ 2, j = 0 \bmod 4. \end{cases}$

ix. $c_7\left(v_8^{(j)}v_8^{(j+1)}\right) = \begin{cases} 5, j = 1 \bmod 4, \\ 1, j = 2 \bmod 4, \\ 2, j = 3 \bmod 4, \\ 3, j = 0 \bmod 4. \end{cases}$

x. $c_7\left(v_9^{(j)}v_9^{(j+1)}\right) = \begin{cases} 1, j = 1 \bmod 4, \\ 2, j = 2 \bmod 4, \\ 3, j = 3 \bmod 4, \\ 4, j = 0 \bmod 4. \end{cases}$

By looking of all possibilities of every geodesic path should be rainbow for distance two, we cannot have less colors. Thus, we can conclude that for $n \geq 8$, $lsrc_4(P_m \times C_n) = \begin{cases} 4; m = 3{,}4; 4|n, \\ 5; m > 4; 4|n, \\ 5;\ 4 \nmid n. \end{cases}$ ■

Figure 3 shows the example of the local strong rainbow coloring of generalized prism for $d = 4$.

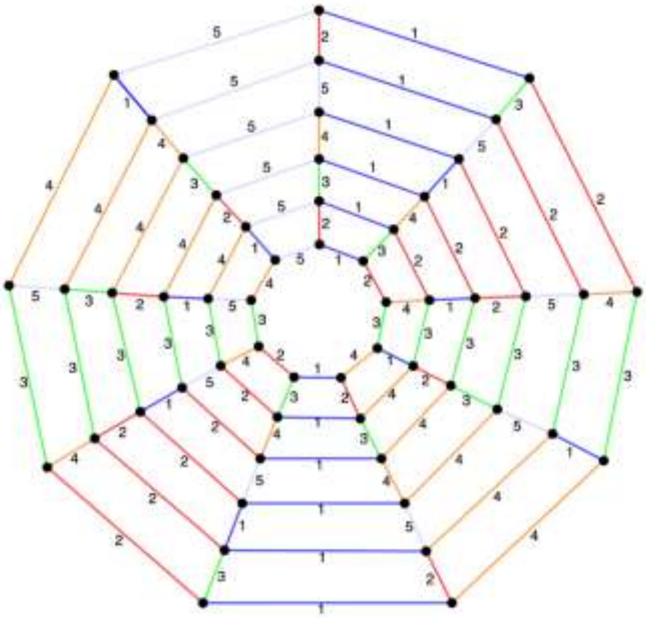

Figure 3. Example of 4-local strong rainbow coloring on prism graph $P_6 \times C_8$

The next theorems are considering the generalized antiprism. Looking at the construction, generalized antiprism can be constructed from generalized prism by adding one edge so that all vertices have degree four. However, the coloring is not that obvious.

Theorem 9

For $n \geq 4$ *and* $m \geq 2$, $lsrc_2\left(A_n^{(m)}\right) = \begin{cases} 2, m \leq 3; n\ even, \\ 3, m \geq 4; n\ even, \\ 3, \qquad n\ odd. \end{cases}$

Proof

Let $G = A_n^{(m)}$, where $V(G) = \left\{v_i^{(j)} \middle| i = 1,2,3, \dots, n; j = 1,2,3, \dots, m\right\}$

$E(G) = \left\{v_i^{(j)} v_{i+1}^{(j)} \middle| i = 1,2,3, \dots, n; j = 1,2,3, \dots, m\right\} \cup \left\{v_i^{(j)} v_i^{(j+1)} \middle| i = 1,2,3, \dots, n; j = 1,2,3, \dots, m-1\right\} \cup \left\{v_i^{(j)} v_{i+1}^{(j+1)} \middle| i = 1,2,3, \dots, n-1; j = 1,2,3, \dots, m-1\right\}$.

Consider the following cases.

Case 1: n even, $m \leq 3$

To show that $lsrc_2\left(A_n^{(m)}\right) = 2$, define the edge coloring $f_1 : E(G) \to \{1,2,3\}$, as follows:

i. $f_1\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 1$, i odd and $j = 1,2,3, \dots, m$.

ii. $f_1\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 2$, $i \in \{2,4,6, \dots, n-2\}$ and $j = 1,2,3, \dots, m$.

iii. $f_1\left(v_n^{(j)} v_1^{(j)}\right) = 2$, $j = 1,2,3, \dots, m$.

iv. $f_1\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 2, & i \text{ odd and } j \text{ odd}, \\ 1, & i \text{ odd and } j \text{ even}. \end{cases}$

v. $f_1\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, & i \text{ even and } j \text{ odd}, \\ 2, & i \text{ even and } j \text{ even}. \end{cases}$

vi. $f_1\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = \begin{cases} 1, & i \text{ odd}, j \in \{1,2\} \cup \{2p | p \geq 2\}, \\ 3, & i \text{ odd}, \quad j \in \{2p+1 | p \geq 1\}. \end{cases}$

vii. $f_1\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = \begin{cases} 2, & i \text{ even}, j \in \{1,2\} \cup \{2p | p \geq 2\}, \\ 3, & i \text{ even}, \quad j \in \{2p+1 | p \geq 1\}. \end{cases}$

viii. $f_1\left(v_n^{(j)} v_1^{(j+1)}\right) = \begin{cases} 2, j \in \{1,2\} \cup \{2p | p \geq 2\}, \\ 3, \quad j \in \{2p+1 | p \geq 1\}. \end{cases}$

Case 2: n even, $m \geq 4$

To show that $lsrc_2(A_n^{(m)}) = 3$, define the edge coloring $f_2 : E(G) \to \{1,2,3\}$, as follows:

i. $f_2\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 1,\ i \in \{1,3,5,\dots,n-2\}$ and $j = 1,2,3,\dots,m$.

ii. $f_2\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 2$, i even and $j = 1,2,3,\dots,m$.

iii. $f_2\left(v_n^{(j)} v_1^{(j)}\right) = 3$, $j = 1,2,3,\dots,m$.

iv. $f_2\left(v_1^{(j)} v_1^{(j+1)}\right) = \begin{cases} 1, & j \text{ odd}, \\ 2, & j \text{ even}. \end{cases}$

v. $f_2\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, & j \text{ odd} \\ 3, & j \text{ even} \end{cases}$, where $i \in \{3,5,\dots,n-2\}$.

vi. $f_2\left(v_i^{(j)} v_i^{(j)}\right) = \begin{cases} 2, & i \text{ even and } j \text{ odd}, \\ 3, & i \text{ even and } j \text{ even}. \end{cases}$

vii. $f_2\left(v_n^{(j)} v_n^{(j)}\right) = \begin{cases} 3, & j \text{ odd}, \\ 1, & j \text{ even}. \end{cases}$

viii. $f_2\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = 1$, i odd and $j = 1,2,3,\dots,m$.

ix. $f_2\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = 2$, i even and $j = 1,2,3,\dots,m$.

x. $f_2\left(v_n^{(j)} v_1^{(j+1)}\right) = 3$, $j = 1,2,3,\dots,m$.

Case 3: n odd

To show that $lsrc_2(A_n^{(m)}) = 3$, define the edge coloring $f_4 : E(G) \to \{1,2,3\}$, as follows:

i. $f_4\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 1$, $i \in \{1,3,5,\dots,n-2\}$ and $j = 1,2,3,\dots,m$.

ii. $f_4\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 2$, i even and $j = 1,2,3,\dots,m$ and $j = 1,2,3,\dots,m$.

iii. $f_4\left(v_n^{(j)} v_1^{(j)}\right) = 3$, $j = 1,2,3,\dots,m$.

iv. $f_4\left(v_1^{(j)} v_1^{(j+1)}\right) = \begin{cases} 1, & j \text{ odd}, \\ 2, & j \text{ even}. \end{cases}$

v. $f_4\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, & j \text{ odd} \\ 3, & j \text{ even} \end{cases}$, where $i \in \{3,5,\dots,n-2\}$.

vi. $f_4\left(v_i^{(j)} v_i^{(j)}\right) = \begin{cases} 2, & i \text{ even and } j \text{ odd}, \\ 3, & i \text{ even and } j \text{ even}. \end{cases}$

vii. $f_4\left(v_n^{(j)} v_n^{(j)}\right) = \begin{cases} 3, & j \text{ odd}, \\ 1, & j \text{ even}. \end{cases}$

viii. $f_4\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = 1$, i odd and $j = 1,2,3,\dots,m$.

ix. $f_4\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = 2$, i even and $j = 1,2,3,\dots,m$.

x. $f_4\left(v_n^{(j)} v_1^{(j+1)}\right) = 3$, $j = 1,2,3,\dots,m$.

By looking of all possibilities of every geodesic path should be rainbow for distance two we cannot have less colors, then we can conclude that for ≥ 4 and $m \geq 2$, $lsrc_2(A_n^{(m)}) = \begin{cases} 2, m \leq 3; n \text{ even}, \\ 3, m \geq 4; n \text{ even} \\ 3, \qquad n \text{ odd}. \end{cases}$, ■

Figure 4 shows the example of the local strong rainbow coloring of generalized antiprism for $d = 2$.

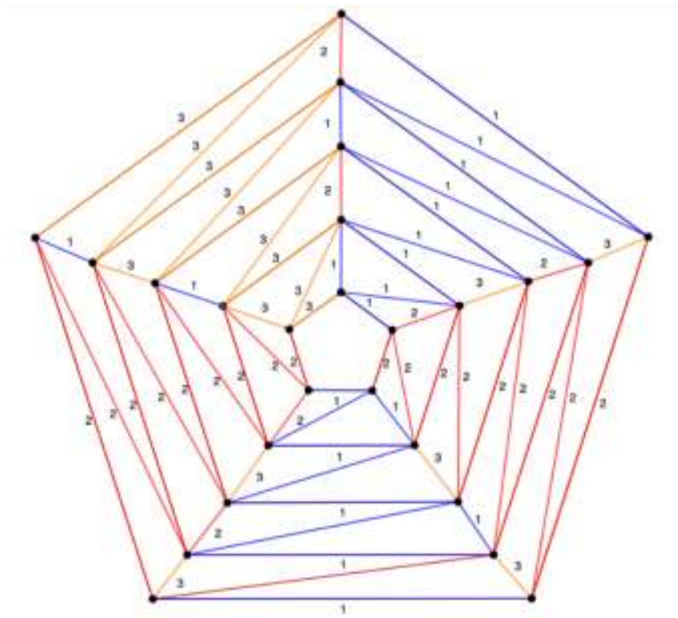

Figure 4. Example of 2-local strong rainbow coloring on antiprism graph $A_5^{(5)}$

Theorem 10

For $n \geq 6$ *and* $m \geq 2$, $lsrc_3\left(A_n^{(m)}\right) = \begin{cases} 3, & m \leq 4; 3|n, \\ 4, & m > 4; 3|n, \\ 4, m \geq 2; & 3 \nmid n. \end{cases}$

Proof

Consider the following cases.

Case 1: $3|n$

To show that $lsrc_3\left(A_6^{(m)}\right) = \begin{cases} 3, m \leq 4 \\ 4, m > 4 \end{cases}$, define the edge coloring $f_5 : E(G) \rightarrow \{1,2,3\}$, as follows:

i. $f_5\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 1, i = 1 \bmod 3$ and $j = 1,2,3, \dots, m$.

ii. $f_5\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 2, i = 2 \bmod 3$ and $j = 1,2,3, \dots, m$.

iii. $f_5\left(v_i^{(j)} v_{i+1}^{(j)}\right) = 3, i = 0 \bmod 3$ and $j = 1,2,3, \dots, m$.

iv. $f_5\left(v_6^{(j)} v_1^{(j)}\right) = 3, j = 1,2,3, \dots, m$.

v. $f_5\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 3, & i = 1 \bmod 3 \text{ and } j = 1, \\ 2, & i = 1 \bmod 3 \text{ and } j = 2 \bmod 3, \\ 1, & i = 1 \bmod 3 \text{ and } j = 0 \bmod 3, \\ 4, & i = 1 \bmod 3 \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

vi. $f_5\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, & i = 2 \bmod 3 \text{ and } j = 1, \\ 3, & i = 2 \bmod 3 \text{ and } j = 2 \bmod 3, \\ 2, & i = 2 \bmod 3 \text{ and } j = 0 \bmod 3, \\ 4, & i = 2 \bmod 3 \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

vii. $f_5\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 2, & i = 0 \bmod 3 \text{ and } j = 1, \\ 1, & i = 0 \bmod 3 \text{ and } j = 2 \bmod 3, \\ 3, & i = 0 \bmod 3 \text{ and } j = 0 \bmod 3, \\ 4, & i = 0 \bmod 3 \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

viii. $f_5\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = 1, i = 1,4$ and $j = 1,2,3, \dots, m$.

ix. $f_5\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = 2, \ i = 2,5$ and $j = 1,2,3, \dots, m$.

x. $f_5\left(v_3^{(j)} v_4^{(j+1)}\right) = 3, \ j = 1,2,3, \dots, m$.

xi. $f_5\left(v_6^{(j)} v_1^{(j+1)}\right) = 3, \ j = 1,2,3, \dots, m$.

Case 2: $3 \nmid n$

To show that $lsrc_3\left(A_n^{(m)}\right) = 4$, we construct $f_6 : E(G) \rightarrow \{1,2,3,4\}$, as follows:

i. $f_6\left(v_i^{(j)}v_{i+1}^{(j)}\right) = 1, i = 1 \bmod 3, i \neq n$ and $j = 1,2,3, \dots, m$.

ii. $f_6\left(v_i^{(j)}v_{i+1}^{(j)}\right) = 2,\ i = 2 \bmod 3$ and $j = 1,2,3, \dots, m$.

iii. $f_6\left(v_i^{(j)}v_{i+1}^{(j)}\right) = 3,\ i = 0 \bmod 3$ and $j = 1,2,3, \dots, m$.

iv. $f_6\left(v_n^{(j)}v_1^{(j)}\right) = 4, j = 1,2,3, \dots, m$.

v. $f_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 1,\ i = 1 \bmod 3, i \neq n \text{ and } j = 1, \\ 2, i = 1 \bmod 3, i \neq n \text{ and } j = 2 \bmod 3, \\ 3, i = 1 \bmod 3, i \neq n \text{ and } j = 0 \bmod 3, \\ 4,\ i = 1 \bmod 3, i \neq n \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

vi. $f_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 2,\ i = 2 \bmod 3 \text{ and } j = 1, \\ 3, i = 2 \bmod 3 \text{ and } j = 2 \bmod 3, \\ 4,\ i = 2 \bmod 3 \text{ and } j = 0 \bmod 3, \\ 1,\ i = 2 \bmod 3 \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

vii. $f_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 3,\ i = 0 \bmod 3 \text{ and } j = 1, \\ 1,\ i = 0 \bmod 3 \text{ and } j = 2 \bmod 3, \\ 4, i = 0 \bmod 3 \text{ and } j = 0 \bmod 3, \\ 2,\ i = 0 \bmod 3 \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

viii. $f_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 1,\ i = 1 \bmod 3 \text{ and } j = 1, \\ 2, i = 1 \bmod 3 \text{ and } j = 2 \bmod 3, \\ 4, i = 1 \bmod 3 \text{ and } j = 0 \bmod 3, \\ 3,\ i = 1 \bmod 3 \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

ix. $f_6\left(v_i^{(j)}v_i^{(j+1)}\right) = \begin{cases} 3,\ i = 0 \bmod 3 \text{ and } j = 1, \\ 4,\ i = 0 \bmod 3 \text{ and } j = 2 \bmod 3, \\ 1, i = 0 \bmod 3 \text{ and } j = 0 \bmod 3, \\ 2,\ i = 0 \bmod 3 \text{ and } j = 1 \bmod 3, j \neq 1. \end{cases}$

x. $f_6\left(v_n^{(j)}v_n^{(j+1)}\right) = \begin{cases} 4,\ j = 1, \\ 1,\ j = 2 \bmod 3, \\ 2,\ j = 0 \bmod 3, \\ 3,\ j = 1 \bmod 3, j \neq 1. \end{cases}$

xi. $f_6\left(v_i^{(j)}v_{i+1}^{(j+1)}\right) = 1,\ i = 1 \bmod 3$ and $j = 1,2, \dots, m$.

xii. $f_6\left(v_i^{(j)}v_{i+1}^{(j+1)}\right) = 2,\ i = 2 \bmod 3$ and $j = 1,2, \dots, m$.

xiii. $f_6\left(v_i^{(j)}v_{i+1}^{(j+1)}\right) = 3,\ i = 0 \bmod 3$ and $j = 1,2, \dots, m$.

xiv. $f_6\left(v_7^{(j)}v_1^{(j+1)}\right) = 4, j = 1,2, \dots, m$.

By looking of all possibilities of every geodesic path should be rainbow for distance two we cannot have less colors, then we can conclude that for $n \geq 6$ and $m \geq 2$, $lsrc_3\left(A_n^{(m)}\right) = \begin{cases} 3, & m \leq 4; 3|n, \\ 4, & m > 4; 3|n, \\ 4, m \geq 2;\ 3 \nmid n. \end{cases}$ ■

Figure 5 shows the example of the local strong rainbow coloring of generalized prism for $d = 3$.

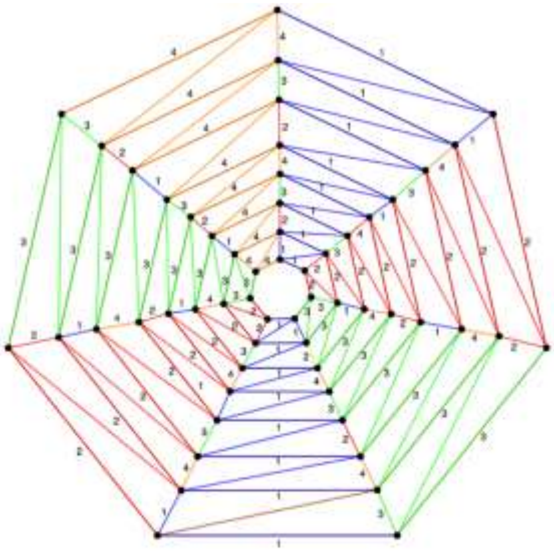

Figure 5. Example of 3-local strong rainbow coloring on antiprism graph $A_7^{(8)}$

Theorem 11

For $n \geq 3$, $\frac{n}{2} \geq 4$ and $m \geq 2$, $lsrc_4\left(A_n^{(m)}\right) = \begin{cases} 4, & m \leq 4; 4|n, \\ 5, & m > 4; 4|n, \\ 5, m \geq 2; & 4 \nmid n. \end{cases}$

Proof

Consider the following cases.

Case 1: $4|n$

We use the smallest number of n which will lead us to the generalized form. To show that $lrsc_4\left(A_8^{(m)}\right) = 4$ define the edge coloring $f_7: E(G) \rightarrow \{1,2,3,4,5\}$, as follows:

i. $f_7\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 1, & i = 1,5, \\ 2, & i = 2,6, \\ 3, & i = 3,7, \\ 4, & i = 4, \end{cases}$ where $j = 1, 2, 3, \ldots, m$.

ii. $f_7\left(v_8^{(j)} v_1^{(j)}\right) = 4,\ j = 1,2,3, \ldots, m,$

iii. $f_7\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 2, i = 1,5; j = 1 \bmod 4, \\ 3, i = 1,5; j = 2 \bmod 4, \\ 4, i = 1,5; j = 3 \bmod 4, \\ 5, i = 1,5; j = 0 \bmod 4. \end{cases}$

iv. $f_7\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 3, i = 2,6; j = 1 \bmod 4, \\ 4, i = 2,6; j = 2 \bmod 4, \\ 1, i = 2,6; j = 3 \bmod 4, \\ 5, i = 2,6; j = 0 \bmod 4. \end{cases}$

v. $f_7\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 4, i = 3,7; j = 1 \bmod 4, \\ 1, i = 3,7; j = 2 \bmod 4, \\ 2, i = 3,7; j = 3 \bmod 4, \\ 5, i = 1,5; j = 0 \bmod 4. \end{cases}$

vi. $f_7\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 1, i = 4,8; j = 1 \bmod 4, \\ 2, i = 4,8; j = 2 \bmod 4, \\ 3, i = 4,8; j = 3 \bmod 4, \\ 5, i = 4,8; j = 0 \bmod 4. \end{cases}$

vii. $f_7\left(v_i^{(j)} v_{i+1}^{(j+1)}\right) = \begin{cases} 1, & i = 1,5, \\ 2, & i = 2,6, \\ 3, & i = 3,7, \\ 4, & i = 4, \end{cases}$ where $j = 1, 2, 3, \ldots, m$.

viii. $f_7\left(v_8^{(j)} v_1^{(j+1)}\right) = 4,\ j = 1,2,3, \ldots, m$

Case 2: $4 \nmid n$

To show that $lrsc_4(P_m \times C_9) = 5$, define the edge coloring $f_8: E(G) \to \{1,2,3,4,5\}$ as follows:

i. $f_8\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 1, & i = 1, 5, \\ 2, & i = 2,6, \\ 3, & i = 3,7, \\ 4, & i = 4,8, \end{cases}$ where $j = 1, 2, 3, \dots, m$.

ii. $f_8\left(v_9^{(j)} v_1^{(j)}\right) = 5, j = 1,2,3, \dots, m$.

iii. $f_8\left(v_i^{(j)} v_i^{(j+1)}\right) = \begin{cases} 2, i = 1,5; j = 1 \bmod 4, \\ 3, i = 1,5; j = 2 \bmod 4, \\ 4, i = 1,5; j = 3 \bmod 4, \\ 5, i = 1,5; j = 0 \bmod 4. \end{cases}$

iv. $f_8\left(v_2^{(j)} v_2^{(j+1)}\right) = \begin{cases} 3, j = 1 \bmod 4, \\ 4, j = 2 \bmod 4, \\ 1, j = 3 \bmod 4, \\ 5, j = 0 \bmod 4. \end{cases}$

v. $f_8\left(v_3^{(j)} v_3^{(j+1)}\right) = \begin{cases} 4, j = 1 \bmod 4, \\ 1, j = 2 \bmod 4, \\ 2, j = 3 \bmod 4, \\ 5, j = 0 \bmod 4. \end{cases}$

vi. $f_8\left(v_4^{(j)} v_4^{(j+1)}\right) = \begin{cases} 1, j = 1 \bmod 4, \\ 2, j = 2 \bmod 4, \\ 3, j = 3 \bmod 4, \\ 5, j = 0 \bmod 4. \end{cases}$

vii. $f_8\left(v_6^{(j)} v_6^{(j+1)}\right) = \begin{cases} 3, j = 1 \bmod 4, \\ 4, j = 2 \bmod 4, \\ 5, j = 3 \bmod 4, \\ 1, j = 0 \bmod 4. \end{cases}$

viii. $f_8\left(v_7^{(j)} v_7^{(j+1)}\right) = \begin{cases} 4, j = 1 \bmod 4, \\ 5, j = 2 \bmod 4, \\ 1, j = 3 \bmod 4, \\ 2, j = 0 \bmod 4. \end{cases}$

ix. $f_8\left(v_8^{(j)} v_8^{(j+1)}\right) = \begin{cases} 5, j = 1 \bmod 4, \\ 1, j = 2 \bmod 4, \\ 2, j = 3 \bmod 4, \\ 3, j = 0 \bmod 4. \end{cases}$

x. $f_8\left(v_9^{(j)} v_9^{(j+1)}\right) = \begin{cases} 1, j = 1 \bmod 4, \\ 2, j = 2 \bmod 4, \\ 3, j = 3 \bmod 4, \\ 4, j = 0 \bmod 4. \end{cases}$

xi. $f_8\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 1, & i = 1, 5, \\ 2, & i = 2,6, \\ 3, & i = 3,7, \\ 4, & i = 4,8, \end{cases}$ where $j = 1, 2, 3, \dots, m$.

xii. $f_8\left(v_9^{(j)} v_1^{(j)}\right) = 5, j = 1,2,3, \dots, m$.

By looking of all possibilities of every geodesic path should be rainbow for distance two, we cannot have less colors. Then we can conclude that for $n \geq 3, \frac{n}{2} \geq 4$ and $m \geq 2$, $lsrc_4\left(A_n^{(m)}\right) = \begin{cases} 4, & m \leq 4; 4|n, \\ 5, & m > 4; 4|n, \\ 5, m \geq 2; & 4 \nmid n. \end{cases}$ ■

Figure 6 shows the example of the local strong rainbow coloring of generalized antiprism for $d = 4$.

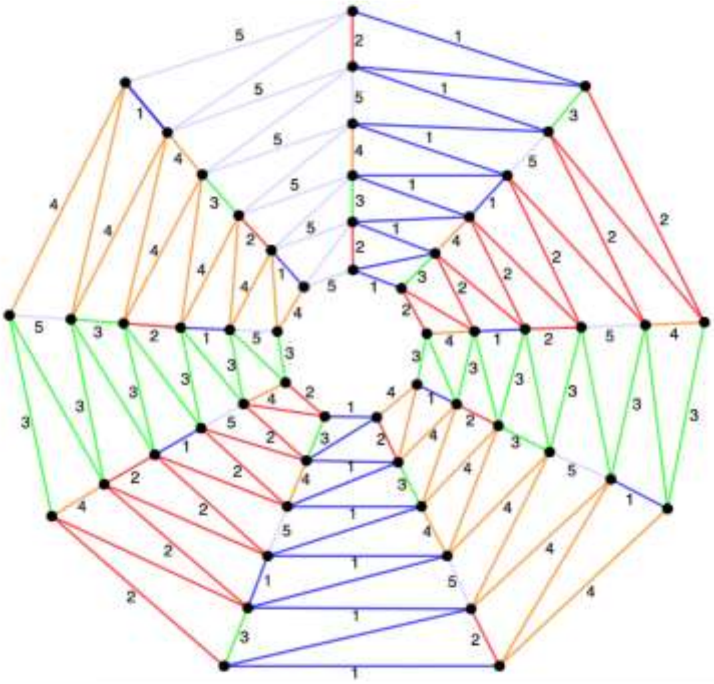

Figure 6. Example of 4-local strong rainbow coloring on antiprism graph $A_9^{(6)}$

4. Conclusions

In this paper, we have the $lsrc_d$ for generalized prism graphs $(P_m \times C_n)$ and generalized antiprism graphs $A_n^{(m)}$, with $d = 2$, $d = 3$ and $d = 4$.

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