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ANALYSIS OF NONLINEAR DISCRETE DYNAMIC SYSTEMS IN THE COURNOT DUOPOLY MODEL WITH THE INFLUENCE OF PROMOTION COSTS

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Abstract
In the field of economics, Cournot duopoly is part of an oligopoly market structure consisting of two firms that have dominant control over a market. Competition between the two firms occurs to determine the optimal quantity of goods to maximize profits. Therefore, an appropriate market strategy is needed to achieve the company's targets. Based on this issue, mathematical modeling is used to analyze the relationship between the two companies. One important factor in introducing a product is promotional activities. Promotions are carried out as a company strategy to attract consumer attention. Then, if there is an increase in demand for goods from consumers, the company will incur marginal costs. Thus, research is conducted that discusses a discrete Cournot duopoly model in predicting the optimal quantity of goods, with the influence of promotional costs and marginal costs in setting market strategies. The research uses quantitative methods in the form of literature studies and analysis of discrete dynamic systems, which includes determining equilibrium points, local stability analysis of equilibrium points, and numerical simulation of the discrete Cournot duopoly model.

Keywords: marginal cost, promotion cost, discrete dynamics, Cournot duopoly.

ANALISIS SISTEM DINAMIK DISKRET NONLINEAR PADA MODEL DUOPOLI COURNOT DENGAN PENGARUH BIAYA PROMOSI

Abstrak
Pada bidang ekonomi, duopoli Cournot merupakan bagian dari struktur pasar oligopoli yang terdiri atas dua perusahaan serta memiliki kontrol dominan pada sebuah pasar. Persaingan antara dua perusahaan terjadi untuk menentukan jumlah barang yang optimal demi mencapai keuntungan maksimal. Oleh sebab itu, dibutuhkan strategi pasar yang tepat agar target perusahaan dapat terealisasi. Berdasarkan permasalahan tersebut, digunakan pemodelan matematika untuk menganalisis keterkaitan antara dua perusahaan. Salah satu faktor penting yang harus dilakukan dalam memperkenalkan barang produksi yaitu kegiatan promosi. Kegiatan promosi dilakukan sebagai strategi perusahaan untuk menarik perhatian konsumen. Kemudian, apabila tejadi kondisi dimana terdapat peningkatan permintaan barang dari konsumen, maka perusahaan akan mengeluarkan biaya marginal. Dengan demikian, dilakukan penelitian yang membahas model diskret duopoli Cournot dalam memprediksi jumlah barang yang optimal dengan pengaruh biaya promosi dan biaya marginal dalam penetapan strategi pasar. Penelitian dilakukan dengan menggunakan metode kuantitatif berupa studi literatur dan analisis sistem dinamik diskret yang terdiri dari penentuan titik kesetimbangan, analisis kestabilan lokal titik kesetimbangan, dan simulasi numerik model diskret duopoli Cournot.

Kata Kunci: biaya marginal, biaya promosi, dinamik diskret, duopoli Cournot.

1. Introduction

Mathematical modeling is one of the techniques used to present a complex system in a mathematical model. Thus, the formulated mathematical model is expected to explain the complex situations observed. A mathematical model consist of variables, parameters, and functions that express the relationships between variables and parameters (Ndii, 2018). Mathematical model have been applied in various fields of science, such as economics, physics, and biology. The mathematical model applied must be suitable for the field in which it is used (Simanihuruk et al., 2023).

The word "economics" originates from Greek words "*oikos*" that means household and "*nomos*" which means rule of management. Economics is a field of science that can provides solutions to the essential needs of human life through the development of all economics resources based on certain principles and theories within an economics system that is deemed effective and efficient. Mathematical modeling in economics is the process of creating mathematical representations of an economic system aimed at understanding and predicting the behaviour of the economic system, specifically how individuals make decisions and what influences those decisions (Rahmatullah, 2018).

Finance is one of the fields of economics that indicate how individuals and businesses acquire and manage money. In this context, economic mathematics uses mathematical concepts to analyse, understand, and make decisions related to financial aspects. The application of economic mathematics involves the use of formulas, equations, and mathematical model to calculate and predict various existing aspects, such as financial planning, investment, loans, and risk (Yusuf et al., 2023).

In the field of economics, market is a place or situation that brings two conditions: the demand from consumers and the offer from producers for every type of goods, services, and resources. Furthermore, market structures can be classified into four different types, namely monopoly, oligopoly, monopolistic competition, and perfect competition (Case & Fair, 2007).

In an oligopoly market, if there are only two companies that have contrl or dominance over the majority of the market share, this condition is known as a duopoly market. The main emphasis in a duopoly market lies in the high level of interdependence between the two companies. For the example, a duopoly market in Indonesia is Indofood and Wings Food for the instant noodle. Additionally, at the international level, there are Cola-Cola and Pepsi as an example for carbonated beverange (Panorama, 2016).

The duopoly market structure has several price formation models, including the Cournot model, Bertrand model, Stackelberg model, and Chamberlin model. In the Cournot model, there is an assumption that the goods produced are homogeneous (Rondhi & Aji, 2015). One important aspect of the Cournot model is the existence of a reaction function. Reaction function shows the quantity of goods a company will produce to maximize its profit, given the quantity produced by the other company (KPPU, 2010).

One crucial factor to consider when developing a market strategy is promotion. Promotion is an activity undertaken to introduce products or services to consumers to maximize profits. Therefore, promotional costs must be allocated to ensure the smooth execution of promotional activities and the success of sales (Firmansyah, 2020).

If promotional activities are successfully implemented, there is a possibility that the company will receive requests for increased production from consumes. Consequently, marginal costs will arise. Marginal costs are costs that adjust according to the production level of a good, creating a difference in fixed costs as a result of the addition of extra production units. Therefore, marginal costs are also an important factor that service providers must consider when trying to optimize profits and design pricing schemes based on utility functions (Nabilla & Nursanti, 2024).

Several applications of the Cournot duopoly model have been discussed by researchers. Askar et al. (2015) analyzed a Cournot duopoly model with similar strategies, Andaluz et al. (2023) discussed a Cournot duopoly model with a Corporate Social Responsibility (CSR) approach, while Zhou et al. (2020) analyzed a Cournot duopoly model with research and development efforts. Meanwhile, Grisakova & Stetka (2022) discussed a Cournot duopoly model with different production outputs and Elsadany (2015) analyzed a Cournot duopoly model with different strategies. Based on the aforementioned discussion, this research examines the stability analysis of the equilibrium point and numerical simulation of a discrete Cournot duopoly model, which is an extension of Elsadany (2015) model. The model is modified by adding the first company's product promotion cost in determining the market strategy and applying different parameter values for promotion costs and marginal costs. Therefore, this research is titled "**Analysis of Nonlinear Discrete Dynamic Systems in the Cournot Duopoly Model with the Influence of Promotion Cost**".

1.1 Dynamic System

A dynamic system is a collection of states of a system whose conditions are constantly changing over time with certain rules and future conditions can be determined if given the present or past conditions (Nagle et al., 2012). Generally, dynamic systems are classified into two types: continuous-time dynamic systems and discrete-time dynamic systems. A system is considered a continuous-time dynamic system if it is applied over a continuous time or within a specific time interval, while a sysrem is considered a discrete-time dynamic system if it is applied at discrete times (Alligood et al., 2000).

1.2 Discrete Dynamic System

Generally, difference equations describe conditions that change over discrete time intervals. For example, given a discrete set, the $n+1$-th set is a function of the n-th set, so a discrete-time dynamic system represented by a single equation can be expressed by the following difference equation:

$$x(n+1) = f\big(x(n)\big) \tag{1.1}$$

If the initial condition is $x(0) = x_0$, then Equation (1.1) forms a sequence

$$x_0, f(x_0), f\big(f(x_0)\big), f\Big(f\big(f(x_0)\big)\Big), \dots$$

Next, the notation

$$f^2(x_0) = f\big(f(x_0)\big), f^3(x_0) = f\Big(f\big(f(x_0)\big)\Big), \dots$$

is used with $f(x_0)$ being the first iterate of f, $f^2(x_0)$ the second iterate of f, and in general $f^n(x_0)$ the n-th iterate of f. Let the set $\{f^n(x_0): n \in \mathbb{N} \cup \{0\}\}$ with $f^0(x_0) = x_0$ be called the orbit of x_0 and denoted by $O(x_0$. The iterated process is an example of a discrete dynamical system. Therefore, Equation (1.1) represents a discrete dynamical system (Elaydi, 2005).

1.3 Promotion Costs

Promotion costs refer to the amount of funds allocated by a company to promote its products in order to increase sales. Promotion can be defined as the activities undertaken to communicate the advantage of a product or service to advance a business (Rustami et al., 2014).

1.4 Cournot Duopoly

Cournot duopoly was first introduced in 1838 by Antoine Augustin Cournot, a French mathematician and economist. Cournot's ideas were presented in a book titled "Researches Into The Mathematical Principles of The Theory of Wealth", which states that the Cournot duopoly is a duopoly model with a primary focus on quantity analysis. In the Cournot duopoly model, it is assumed that two companies simultaneously seeking to maximaze their profits plan their marketing strategies regarding the quantity of a homogeneous good to be sold in a specific time period (Raharjo, 2020).

However, in reality, profits do not solely depend on a company's own marketing strategy but are also influenced by the strategies adopted by other companies, as market prices are a function of total quantity. To address this issue, the concept of arbitrage is introduced. In economics, arbitrage refers to the simultaneous buying and selling of the same or similar assets in different markets to profit from slight discrepancies in the asset's stated price. Due to these pricing differences, profits can be realized (Buchanan, 2012). Consequently, it is assumed that the first company acts as an arbitrageur while the second company takes on the role of a follower (Panorama, 2016).

1.5 A Mathematical Model of Cournot Duopoly with Different Strategies

Elsadany (2015) in his research titled "A Dynamic Cournot Duopoly Model with Different Strategies" has formulated a mathematical model regarding the differences in strategies between two companies in a Cournot duopoly model. The mathematical model of Cournot duopoly can be expressed as follows:

$$x_1^{t+1} = x_1^t + \alpha_1 x_1^t \left(a - c_1 - b\sqrt{x_1^t + x_2^t} - \frac{bx_1^t}{2\sqrt{x_1^t + x_2^t}} \right) \tag{1.2}$$

$$x_2^{t+1} = \frac{2(a-c_2)\sqrt{x_1^t + x_2^t} - 2bx_1^t - bx_2^t}{2b} \tag{1.3}$$

with the description of variables and parameters presented in the Table 1.

Table 1. Variables and Parameters

No.	Simbol	Keterangan	Satuan
1.	x_1^t	Production quantity of the first company	Unit
2.	x_2^t	Production quantity of the second company	Unit
3.	t	Period	Month
4.	a	Positive constant	-
5.	b	Positive coefficient	-
6.	c_1	Marginal cost of the first company	Cost unit
7.	c_2	Marginal cost of the second company	Cost unit
8.	α_1	Market strategy adjustment speed	Unit/Profit

2. Methods

In this study, the method used is a literature review involving a study of books, scientific journals, teaching materials, and other relevant references on the Cournot duopoly model. The steps used in this research are as follows:

1. Literature Review
   Literature review is the initial step undertaken to gather information from various sources such as books and journals related to the dynamic Cournot duopoly model. This is followed by the collection of supporting concepts necessary for solving the problem, thus providing the fundamental ideas for developing a solution.
2. Modifying the Discrete-Time Dynamic Cournot Duopoly Model
   At this stage, a dynamic Cournot duopoly model is constructed, referring to the model used by Elsadany (2015). The model is modified as follows:
   $$x_1^{t+1} = x_1^t + \alpha_1 x_1^t \left(a - c_1 - d_1 - b\sqrt{x_1^t + x_2^t} - \frac{b x_1^t}{2\sqrt{x_1^t + x_2^t}} \right) \quad (2.1)$$
   $$x_2^{t+1} = \frac{2(a-c_2)\sqrt{x_1^t + x_2^t} - 2b x_1^t - b x_2^t}{2b} \quad (2.2)$$
   where d_1 represents the promotional cost of the first company's product.
3. Analyzing the Discrete Dynamic Cournot Duopoly Model
   The analysis of the discrete dynamic Cournot duopoly model begins by finding the equilibrium points of the model expressed in Equation (1.2) and Equation (1.3). Subsequently, the conditions for the existence of these points are determined and stability conditions are sought by analyzing the eigenvalues of the Jacobian matrix.
4. Numerical Simulation
   Numerical simulaton is conducted on the discrete dynamic Cournot duopoly model using parameter values derived from Elsadany (2015) and assumptions. The parameter values are first tested to ensure they meet the local stability conditions. Subsequently, the model is simulated and run with varying parameter values.
5. Interpretation
   The simulation results obtained from the program output are evaluated to check their consistency with the results of dynamic analysis.

3. Results and Discussion

3.1 Cournot Duopoli Model

In this discussion, the notation for the production quantity of the first company at time $t+1$ is changed to P_{n+1} and the notation for the production quantity of the second company at time $t+1$ is changed to R_{n+1}. Therefore, based on the type of market strategy formation of both company, the two-dimensional system that characterizes the dynamics in the Cournot duopoly economics model is as follows:

$$P_{n+1} = P_n + \alpha_1 P_n \left(a - c_1 - d_1 - b\sqrt{P_n + R_n} - \frac{b P_n}{2\sqrt{P_n + R_n}} \right) \quad (3.1)$$
$$R_{n+1} = \frac{2(a-c_2)\sqrt{P_n + R_n} - 2bP_n - bR_n}{2b} \quad (3.2)$$

with the consideration that the model only examines the trajectories of the system with positive values and that the equilibrium point $(0,0)$ is not defined.

3.2 Equilibrium Points & Local Stability Analysis of the Equilibrium Points

An equilibrium points in Equation (3.1) and Equation (3.2) is the point (P_n^*, R_n^*) that satisfies the conditions

$$f(P_n^*, R_n^*) = P_n^* \quad (3.3)$$
$$g(P_n^*, R_n^*) = R_n^* \quad (3.4)$$

Meanwhile, to determine the stability of the equilibrium points in Equation (3.1) and Equation (3.2), an eigen value analysis of the Jacobian matrix for each equilibrium points is conducted. The Jacobian matrix for Equation (3.1) and Equation (3.2) at the equilibrium points $E(P_n^*, R_n^*)$ is as follows:

$$J(P_n^*, R_n^*) = \begin{bmatrix} \frac{\partial f(P_n^*, R_n^*)}{\partial P} & \frac{\partial f(P_n^*, R_n^*)}{\partial R} \\ \frac{\partial g(P_n^*, R_n^*)}{\partial P} & \frac{\partial g(P_n^*, R_n^*)}{\partial R} \end{bmatrix}$$

with

$$\frac{\partial f(P_n^*, R_n^*)}{\partial P} = 1 + \alpha_1$$
$$\left(a - c_1 - d_1 - b\sqrt{P_n^* + R_n^*} - \frac{bP_n^*}{2\sqrt{P_n^* + R_n^*}}\right)$$
$$+\alpha_1 P_n^* \left(\frac{bP_n^*}{4(P_n^* + R_n^*)^{\frac{3}{2}}} - \frac{b}{\sqrt{P_n^* + R_n^*}}\right)$$

$$\frac{\partial f(P_n^*, R_n^*)}{\partial R} = -\frac{\alpha_1 b P_n^* (P_n^* + 2R_n^*)}{4(P_n^* + R_n^*)^{\frac{3}{2}}}$$
$$\frac{\partial g(P_n^*, R_n^*)}{\partial P} = \frac{a - c_2}{2b\sqrt{P_n^* + R_n^*}} - 1$$
$$\frac{\partial g(P_n^*, R_n^*)}{\partial R} = \frac{a - c_2}{2b\sqrt{P_n^* + R_n^*}} - \frac{1}{2}$$

By assuming that

$$\theta = \alpha_1\left(a - c_1 - d_1 - b\sqrt{p^* + r^*} - \frac{bp^*}{2\sqrt{p^*+r^*}}\right)$$
$$\phi = \alpha_1 p^* \left(\frac{bp^*}{4(p^*+r^*)^{\frac{3}{2}}} - \frac{b}{\sqrt{p^*+r^*}}\right)$$
$$\psi = \frac{\alpha_1 b p^*(p^*+2r^*)}{4(p^*+r^*)^{\frac{3}{2}}}$$
$$\sigma = \frac{a-c_2}{2b\sqrt{p^*+r^*}}$$
$$A = (\theta + \phi)$$
$$B = \left(\sigma - \frac{1}{2}\right)$$
$$C = \psi(\sigma - 1)$$

then the conditions for existence of equilibrium points and the type of stability at the equilibrium points of the system can be summarized in Table 2.

Table 2. Equilibrium Points Conditions and Local Stability Conditions

Equilibrium Points	Existence Conditions	Local Stability Conditions
$E_1 = \left(0, \frac{4(a-c_2)^2}{9b^2}\right)$	$b \neq 0$	Saddle unstable
$E_2 = (p^*, r^*)$	$2a - c_1 - c_2 - d_1 > 0$ $a - 3c_1 + 2c_2 - 3d_1 > 0$ $a + 2c_1 - 3c_2 + 2d_1 > 0$ $b \neq 0$	$\frac{3}{2}A + \psi < \sigma(\theta + \phi + \psi)$ $\frac{3}{2} > \theta B + \phi B + C + \sigma$ $0 < 1 + \frac{1}{2}A + C + \sigma(2 + \theta + \phi)$

3.3 Numerical Simulation

Numerical simulations are conducted to illustrate the dynamics of the discrete Cournot doupoly model based on the existence conditions and local stability conditions at the equilibrium points. The parameter values have been selected and tested, so that they meet the axistence and local stability conditions of the equilibrium points in Table 2. The parameter values used for the numerical simulations are shown in Table 3.

Table 3. Parameter Values in the Numerical Simulation

Parameter	Numerical Simulation I	Numerical Simulation II A	Numerical Simulation II B	Description
α_1	0,6	0,6	0,6	Elsadany, A. A. (2015)
a	5	5	5	Assumption
b	1	1	1	Elsadany, A. A. (2015)
c_1	1	1	1	Elsadany, A. A. (2015)
c_2	1	2	1,8	Assumption
d_1	1	1,5	1,5	Assumption

The initial values used in the numerical simulation are assumed to be as follows:

1. Numerical simulation I and numerical simulation II use initial values $(P_n, R_n) = (1; 0{,}3)$.
2. Numerical simulation III uses initial values $(P_n, R_n) = (2; 0{,}1)$.
3. Numerical simulation IV uses initial values $(P_n, R_n) = (6; 1{,}5)$.

3.3.1 Numerical Simulation I

Numerical simulation I is carried out using the parameter values in Table 3 (Numerical Simulation I). The simulation results are presented in the Figure 1, Figure 2, and Figure 3.

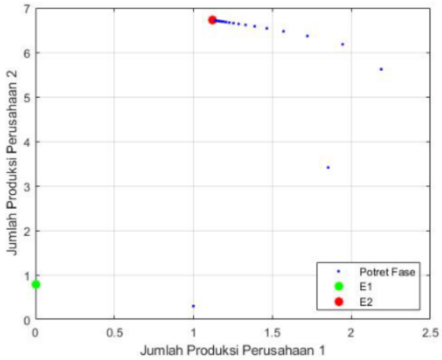

Figure 1. Phases portrait c_2=1 dan d_1=1

Figure 1 shows that the equilibrium points $E_2 = (1{,}12; 6{,}72)$ exist and is asymptotically stable. This can be observed from the direction of the phase potait, which converges toward point E_2. Meanwhile, the equilibrium point $E_1 = 0; 0{,}790123$ is unstable.

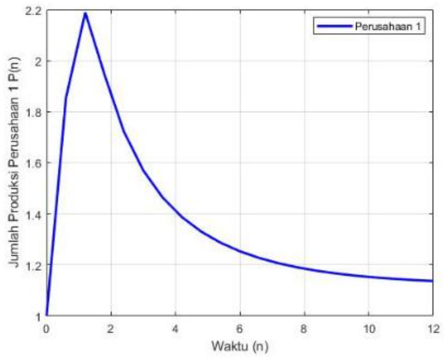

Figure 2 Simulation of the First Company's Production Quantity Pn Over Time n

Figure 2 shows that initially, the production quantity of the first company increases, then over time decreases and stabilizes toward the point 1,13674.

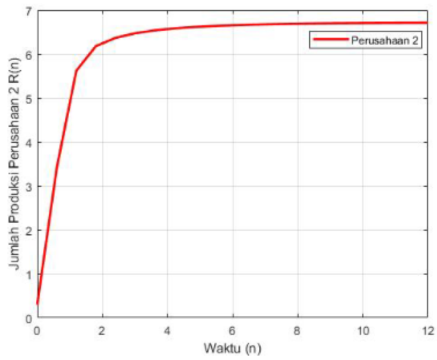

Figure 3 Simulation of the First Company's Production Quantity Rn Over Time n

Figure 3 shows that over time, the production quantity of the second company increases and will stabilize toward the point 6,71208.

3.3.2 Numerical Simulation II

Numerical simulation II is carried out using the parameter values in Table 3 (Numerical Simulation II). The simulation results are presented in the Figure 4, Figure 5, Figure 6, and Figure 7.

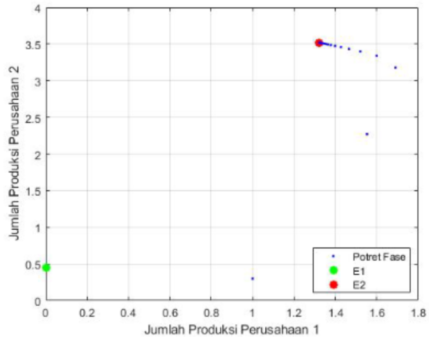

Figure 4 Phases portrait c_2=2 dan d_1=1,5

Figure 4 shows that the equilibrium point $E_2 = (1{,}3218; 3{,}51899)$ exists and is asymptotically stable. This can be observed from the direction of the phase portrait converging toward point E_2. Meanwhile, the equilibrium point $E_1 = (0; 0{,}4444)$ is unstable.

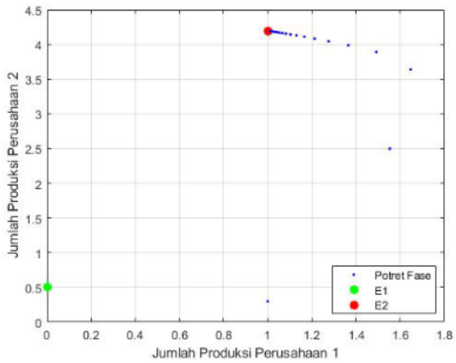

Figure 5 Phases portrait c_2=1,8 dan d_1=1,5

Figure 5 shows that the equilibrium point $E_2 = (1{,}0032; 4{,}1952)$ exists and is asymptotically stable. This can be observed from the direction of the phase portrait converging toward point E_2. Meanwhile, the equilibrium point $E_1 = (0; 0{,}505679)$ is unstable.

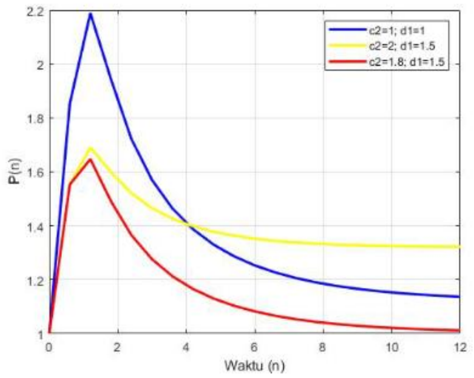

Figure 6 Simulation of the First Company's Production Quantity Pn Over Time n

Figure 6 shows that when the value of the c_2 and d_1 is increased, the production quantity of the first company increases. However, when c_2 is decreased while d_1 remains constant, the production quantity of the first company decreases, reaching a minimum quantity.

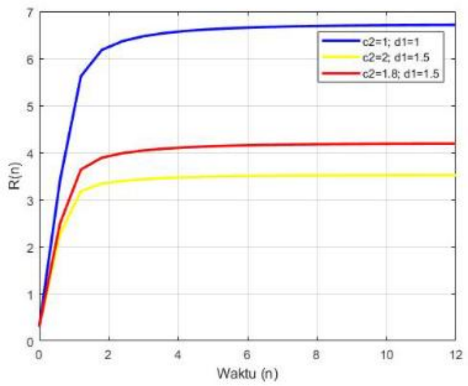

Figure 7 Simulation of the Second Company's Production Quantity Rn Over Time n

Figure 7 shows that when the value of the c_2 and d_1 is increased, the production quantity of the second company decreases. Conversely, when c_2 is decreased while d_1 remains constant, the production quantity of the second company increases. Therefore, it can be concluded that the magnitude of c_2 and d_1 significantly affects the production quantity of both company.

3.3.3 Numerical Simulation III

Numerical simulation III is carried out using the parameter values in Table 3 (Numerical Simulation II A) but with changes made to the value of c_1 , which represents the marginal cost of the first company in order to examine the effect of marginal cost on the quantity of goods produced. The effect of changes in the value of c_1 on the quantity of goods produced is presented in Figure 8 and Figure 9.

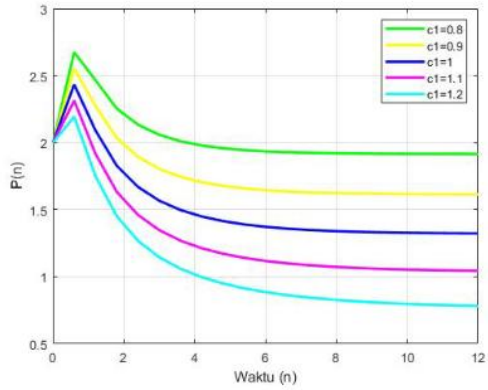

Figure 8 Simulation of the First Company's Production Quantity Pn Over Time (n) with Variations in the Value of c_1

Figure 8 shows the effect of c_1 on the production quantity of the first company. If the value of c_1 increases, the production quantity of the first company initially rises, then decreases until it stabilizes at a certain value.

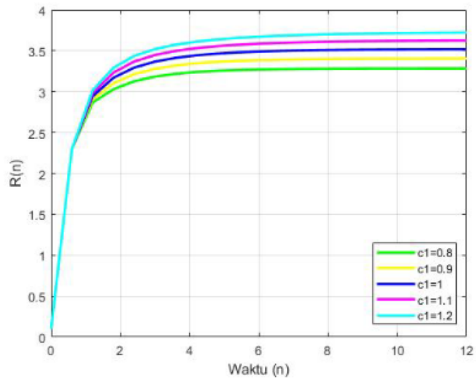

Figure 9 Simulation of the Second Company's Production Quantity Rn Over Time (n) with Variations in the Value of c_1

Figure 9 shows the effect of c_1 on the production quantity of the second company. If the value of c_1 increases, the production quantity of the second company increases until it stabilizes at a certain value. Therefore, it can be concluded that the magnitude of the marginal cost incurred by the first company can impact the production quantity of both company.

3.3.4 Numerical Simulation IV

Numerical simulation IV is carried out using the parameter values in Table 3 (Numerical Simulation II A) but with changes made to the value of d_1 , which represents the promotional cost of the first company in order to examine the effect of promotional cost on the quantity of goods produced. The effect of changes in the value of d_1 on the quantity of goods produced is presented in Figure 10 and Figure 11.

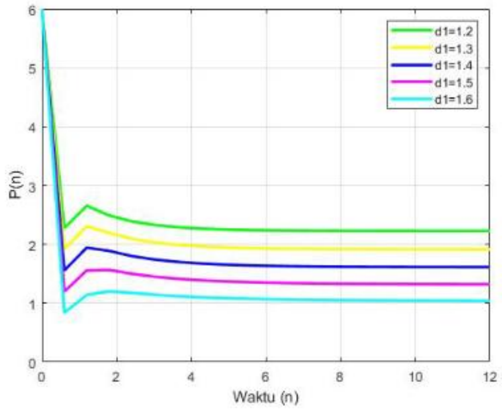

Figure 10 Simulation of the First Company's Production Quantity P_n Over Time (n) with Variations in the Value of d_1

Figure 10 shows the effect of d_1 on the production quantity of the first company. If the value of d_1 increases, the production quantity of the first company initially decreases, then increases, and subsequently decreases again until it stabilizes at a certain value.

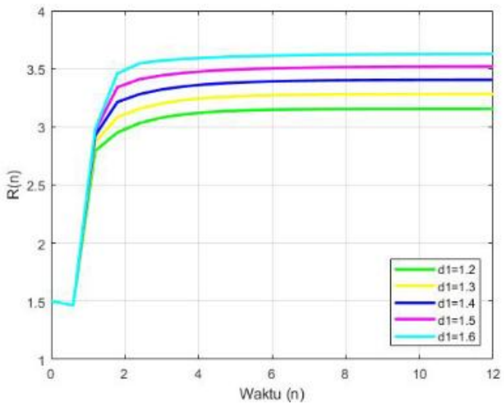

Figure 11 Simulation of the Second Company's Production Quantity R_n Over Time (n) with Variations in the Value of d_1

Figure 11 shows the effect of d_1 on the production quantity of the second company. If the value of d_1 increases, the production quantity of the second company increases until it stabilizes at a certain value. Therefore, it can be concluded that the magnitude of the promotional costs incurred by the first company can impact the production quantity of both company.

4. Conclusion

Based on the results of the research and discussion, the following conclusions can be drawn:

1. The determination of equilibrium points indicates that the studied Cournot duopoly model has two equilibrium points, namely the equilibrium point $E_1(P_n^*, R_n^*) = \left(0, \frac{4(a-c_2)^2}{9b^2}\right)$ which represents that when the first company does not produce goods, the second company continues to produce goods and the equilibrium point $E_2(P_n^*, R_n^*) = (p^*, r^*)$ with $p^* = \frac{4(2a-c_1-c_2-d_1)(a-3c_1+2c_2-3d_1)}{25b^2}$ and $r^* = \frac{4(2a-c_1-c_2-d_1)(a+2c_1-3c_2+2d_1}{25b^2}$ which represents that both companies continue the production process under the condition of the existence of the equilibrium point E_2 namely $2a - c_1 - c_2 - d_1 > 0, a - 3c_1 + 2c_2 - 3d_1 > 0, a + 2c_1 - 3c_2 + 2d_1 > 0,$ and $b \neq 0$.
2. The stability analysis at both equilibrium points of the Cournot duopoly model shows that the equilibrium point $E_1(P_n^*, R_n^*) = \left(0, \frac{4(a-c_2)^2}{9b^2}\right)$ is an unstable saddle point.

   Meanwhile, the equilibrium point $E_2(P_n^*, R_n^*) = (p^*, r^*)$ with $p^* = \frac{4(2a-c_1-c_2-d_1)(a-3c_1+2c_2-3d_1)}{25b^2}$ and $r^* = \frac{4(2a-c_1-c_2-d_1)(a+2c_1-3c_2+2d_1}{25b^2}$ is stable under the local stability conditions at the equilibrium point E_2 as follows $\frac{3}{2}A + \psi < \sigma(\theta + \phi + \sigma), \frac{3}{2} > \theta B + \phi B + C + \sigma,$ and $0 < 1 + \frac{1}{2}A + C + \sigma(2 + \theta + \phi)$.
3. The results of the numerical simulation align with the equilibrium point stability analysis, indicating that the level of promotional cost for the first company or d_1 affects the production quantities of both companies. As d_1 increases, the production quantity of the first company initially decreases, then increases, and subsequently decreases again until stabilizes at a certain value. In contrast, the production quantity of the second company increases and then stabilizes at a certain value.

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