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# The Total Disjoint Irregularity Strength of Double and Triple Star Graphs

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**Abstract:** Combination of the total irregular vertex and the total irregular edge labeling referred to as the total irregular total labeling of $G$ is a function $f: V \cup E \to \{1,2,3,\cdots,k\}$ such that for every two different vertices in $V(G)$ and every two different edges in $E(G)$, their weights are different. Adding a condition where there is no intersection of the vertex-weight set and the edge-weight set, the labeling is called the total disjoint irregular total labeling. In this research, we examined the total irregular total labeling of double star and triple star graphs, which is a class of caterpillar graphs and obtained the exact values of the total disjoint irregularity strength of those graphs corresponding to the lower bound for the total irregularity strength of graph $G$ ($ts(G)$).

2010 Mathematical Subject Classification: 05C78.
**Keywords:** double star graph, total disjoint irregular total labeling, triple star graph.

## 1. Introduction

In everyday life, there are various simple problems that requires solutions resulting in science and technology experiencing many developments. In Mathematics, one of the topics that can solve these problems is graph theory. Graph theory was first used to solve the Konigsberg bridge problem by Leonhard Euler in 1736. One of the interesting topics of graph theory is graph labeling. Graph labeling is a function that maps a vertex set or edge set or both to an integer set, hereinafter referred to as edge labeling, vertex labeling, and total labeling. Edge irregular total labeling and vertex irregular total labeling were introduced by Bača, et al. in 2002 [1].

In [1], Bača *et al.* have provided a lower bound for the total edge irregularity strength ($tes(G)$) and the total vertex irregularity strength ($tvs(G)$) of any graph $G$ and determined the exact value for certain classes of graphs. Many studies were then developed to obtain these exact values in a wider class of graphs or even any graph in general. Marzuki *et al.* [2] then combined the two labels and define a new label, namely totally irregular total labeling. Totally irregular total $k$-labeling $\lambda : V \cup E \to \{1, 2, \cdots, k\}$ of a graph $G$ is a total labeling such that $G$ has a vertex irregular total labeling and an edge irregular total labeling at the same time. The minimum integer $k$ for $G$ to have a totally irregular total $k$-labeling is called the total irregular total strength of $G$, denoted by $ts(G)$. Many research that obtains the result about the exact value of $ts(G)$ obtained for certain classes of graphs shows that this theory is quite developed. In addition, if the value of $ts(G)$ is found, it can be used to answer the problems of the upper limit of both $tes(G)$ and $tvs(G)$.

Determining the exact value of $tes(G)$, $tvs(G)$, and $ts(G)$ for any graph or class of simpler graphs can be quite difficult [3]-[11]. Tilukay *et al.* [12] provided a new labeling whose parameter is the upper limit of the parameters of the labels above. A total disjoint irregular total labeling of a graph $G = (V,E)$, with the edge weight set $W(E)$ and the vertex weight set $W(V)$ on $G$ is a totally irregular total labeling for which $W(E) \cap W(V) = \emptyset$. Its parameter is called the total disjoint irregularity strength of $G$, denoted by $ds(G)$. They [12] have given a lower bound on values $ds(G)$ and has assigned the exact value $ds(G)$ for path graphs, cycle graphs, star graphs, and complete graphs.

**Theorem A** [12]. Let $G = (V,E)$ be a connected graph. Let $v$ be a pendant vertex and $n_i (i = 1,2)$ be the number of vertices of degree $i$. Then

$$ds(G) \geq \begin{cases} \max\left\{n_1, \left\lceil \frac{|E| + n_1 + n_2 + 1}{3} \right\rceil\right\}, & \text{if } v \in V; \\ \left\lceil \frac{|E| + n_1 + n_2 + 2}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

**Theorem B** [12]. Let $P_n$ be a path graph with $n$ vertices, $n \geq 1$, then

$$ds(P_n) = \begin{cases} 3, & \text{for } n = 3; \\ \left\lceil \frac{2n}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

In investigating a totally disjoint irregular total labeling of tree, as posted as an open problem given in [12], we obtained some results for a class of caterpillar. Caterpillar is a tree with the property that the removal of its endpoints leaves a path.

## 2. Results and Discussion

### 2.1. Double Star Graph

Double star graph $S_{m,n}$ is a caterpillar with the center path of order two. Each of both center vertices adjacent to $m$ and $n$ vertices, respectively.

**Lemma 1.** For positive integer $m$ and $n$, $S_{m,n}$ is a double star graph with $m + n$ pendant vertices. Then

$$ds(S_{m,n}) = \begin{cases} 3, & \text{for } m = n = 1; \\ m + n, & \text{otherwise.} \end{cases}$$

Proof:
We divide the proof into two cases as follows.
Case 1. For $m = n = 1$, we have $S_{1,1} \cong P_4$, hence, from Theorem B, it can be concluded that $ds(S_{1,1}) = 3$.
Case 2. For $m \neq n \neq 1$,
Next, for let $V(S_{m,n}) = \{a_i, b_j | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{a, b\}$, where $a$ is a vertex of degree $m$ and $b$ is a vertex of degree $n$. Since $S_{m,n}$ has $m + n$ vertices of degree 1, by Theorem A, it can be obtained that $ds(S_{m,n}) \geq m + n$. Next, to prove that $ds(S_{m,n}) \leq m + n$, we divide into 2 subcases as follows:
Subcase 1. If $m = n$.

We construct a labeling $f_1 : V \cup E \to \{1, 2, \cdots, 2n\}$ from $S_{n,n}$ as follows:

$f_1(a) = f(b) = 2n;$

$$f_1(a_i) = \begin{cases} i, & \text{for } 1 \leq i \leq n-1; \\ 2, & \text{for } i = n; \end{cases}$$

$$f_1(b_i) = \begin{cases} n+i, & \text{for } 1 \leq i \leq n-1; \\ n+1, & \text{for } i = n; \end{cases}$$

$f_1(ab) = 4;$

$$f_1(aa_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq n-1; \\ n-1, & \text{for } i = n; \end{cases}$$

$$f_1(bb_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq n-1; \\ n, & \text{for } i = n. \end{cases}$$

From the construction of $f_1$, it can be seen that the largest label is $f_1(a) = f_1(b) = 2n$. Next, we evaluate the vertex and edge weights of $S_{m,n}$, as follows:

$W_1(a) \quad = 4n + 2;$
$W_1(b) \quad = 4n + 3;$
$W_1(a_i) \quad = i + 1$, for $1 \leq i \leq n$;
$W_1(b_i) \quad = n + i + 1$, for $1 \leq i \leq n$;
$W_1(ab) \ = 4n + 4;$
$W_1(aa_i) = 2n + i + 1$, for $1 \leq i \leq n$;
$W_1(bb_i) = 3n + i + 1$, for $1 \leq i \leq n$.

The vertex-weight set $W_1(V(S_{n,n})) = \{W_1(a_i), W_1(b_i)|1 \leq i \leq n\} \cup \{W_1(a), W_1(b)\} = \{2,3,\cdots,2n+1\} \cup \{4n+2,4n+3\}$ and the edge-weight set $W_1(V(S_{n,n})) = \{W_1(aa_i), W_1(bb_i)|1 \leq i \leq n\} \cup \{W_1(ab)\} = \{2n+2,2n+3,\ldots,4n+4\} \cup \{4n+4\}$ show that there are no two vertices that have the same weight, no two edges have the same weight, and $W_1(V) \cap W_1(E) = \emptyset$.

Subcase 2. If $m \neq n$.

Without loss generality, hereinafter we arrange $m < n$. We construct the labeling $f_2: V \cup E \rightarrow \{1, 2, \cdots, m+n\}$ from $S_{m,n}$ as follows:

$f_2(a) \quad = f_2(b) = m + n;$

$$f_2(a_i) \ = \begin{cases} i, & \text{for } 1 \leq i \leq m-1; \\ 2, & \text{for } i = m; \end{cases}$$

$$f_2(b_j) \ = \begin{cases} m+j, & \text{for } 1 \leq j \leq n-1; \\ n+1, & \text{for } j = n; \end{cases}$$

$f_2(ab) \ = m + n;$

$$f_2(aa_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq m-1; \\ m-1, & \text{for } i = m; \end{cases}$$

$$f_2(bb_i) = \begin{cases} 1, & \text{for } 1 \leq j \leq n-1; \\ m, & \text{for } j = n. \end{cases}$$

From the construction of $f_2$, it can be seen that the largest label is $f_2(a) = f_2(b) = m + n$. Next, the vertex weights and edge weights are obtained as follows:

$W_2(a) \quad = 3(m+n) - n + m - 2;$
$W_2(b) \quad = 3(m+n) - 1;$
$W_2(a_i) \quad = i + 1, \quad$ for $1 \leq i \leq m$;
$W_2(b_j) \quad = m + j + 1, \quad$ for $1 \leq j \leq n$;
$W_2(ab) \ = 3(m+n);$
$W_2(aa_i) = m + n + i + 1, \quad$ for $1 \leq i \leq m$;
$W_2(bb_j) = 2m + n + j + 1,$ for $1 \leq j \leq n$.

The vertex-weight set $W_2(V(S_{m,n})) = \{W_2(a_i), W_2(b_j)|1 \leq i \leq m\} \cup \{W_2(a), W(b) = \{2,3,\cdots,m+n+1\} \cup \{3(m+n)-2, 3(m+n+1)\}$ and the edge-weight set $W_2(E(S_{n,n})) = \{W_2(aa_i), W_2(bb_i)|1 \leq i \leq m, 1 \leq j \leq n\} \cup \{W_2(ab) = \{m+n+2, m+n+3, \cdots, 2m+2n+1\} \cup \{3(m+n)\}$ show that there are no two vertices that have the same weight, no two edges have the same weight, and $W_2(V) \cap W_2(E) = \emptyset$.

From Subcases 1 and 2, we have $f_1$ and $f_2$ are disjoint total irregular total labeling. Thus, $ds(S_{m,n}) = m + n$. It completes the proof. ■

## 2.2. Triple Star Graph

Triple star graph $S_{m,n,q}$ is a caterpillar with the center path of order three. Each of center vertices adjacent to $m$, $n$, and $q$ vertices, respectively.

**Lemma 2.** For positive integer $n$, $S_{n,n,n}$ is a triple star graph with $3n$ pendant vertices. Then

$$ds(S_{n,n,n}) = \begin{cases} 4, & \text{for } n = 1; \\ 3n, & \text{otherwise.} \end{cases}$$

Proof:

Let $V(S_{n,n,n}) = \{a_i, b_i, c_i | 1 \leq i \leq n\} \cup \{a, b, c\}$ where $a$, $b$, and $c$ are vertices of degree $n$. By Theorem A, we have $ds(S_{1,1,1}) \geq 4$ and for $n > 1$, since $(S_{n,n,n})$ have $3n$ vertex of degree 1, $ds(S_{n,n,n}) \geq 3n$. To prove the reverse inequality, we construct the labeling $f_3: V \cup E \to \{1, 2, \cdots, n\}$ from $(S_{n,n,n})$ as follows:

For $n = 1$,

$f_3(a) = f_3(c) = f_3(ab)\ 3;$

$f_3(b) = f_3(bc) = 4;$

$f_3(a_1) = f_3(b_1) = 1 f_3(bb_i) = 1;$

$f_3(c_1) = f_3(aa_i) = f_3(cc_i) = 2.$

Based on the function defined above, it can be seen that the largest label is $f_3(b) = f_3(bc) = 4$. Next, the vertex weights and edge weights are $W_3(a) = 8$, $W_3(b) = 12$, $W_3(c) = 9$, $W_3(a_1) = 3$, $W_3(b_1) = 2$, $W_3(c_1) = 4$, $W_3(ab) = 10$, $W_3(bc) = 11$, $W_3(aa_1) = 6$, $W_3(bb_1) = 5$, and $W_3(cc_1) = 7$. It can be seen that no two vertices have the same weight, no two edges have the same weight, and $W_3(V) \cap W_3(E) = \emptyset$.

For $n > 1$,

$f_3(a) = f_3(b) = f_3(c) = 3n;$

$$f_3(a_i) = \begin{cases} i, & \text{for } 1 \leq i \leq n-1; \\ n+1, & \text{for } i = n; \end{cases}$$

$$f_3(b_i) = \begin{cases} n+i-1, & \text{for } 1 \leq i \leq n-1; \\ n+1, & \text{for } i = n; \end{cases}$$

$$f_3(c_i) = \begin{cases} 2n+i-2, & \text{for } 1 \leq i \leq n-1; \\ n+2, & \text{for } i = n; \end{cases}$$

$$f_3(aa_i) = f_3(cc_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq n-1; \\ 2n-1, & \text{for } i = n; \end{cases}$$

$$f_3(bb_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq n-1; \\ 2n-2, & \text{for } i = n; \end{cases}$$

$f_3(ab) = 4;$

$f_3(bc) = 5.$

Based on the function defined above, it can be seen that the largest label is $f_3(a) = f_3(b) = f_3(c) = 3n$. Next, the vertex weights and edge weights are obtained as follows:

$W_3(a) = 6n + 2;$

$W_3(b) = 6n + 6;$

$W_3(c) = 6n + 3;$

$$W_3(a_i) = \begin{cases} i+1, & \text{for } 1 \leq i \leq n-1; \\ 3n, & \text{for } i = n; \end{cases}$$

$$W_3(b_i) = \begin{cases} n+i, & \text{for } 1 \le i \le n-1; \\ 3n-1, & \text{for } i = n; \end{cases}$$

$$W_3(c_i) = \begin{cases} 2n+i-1, & \text{for } 1 \le i \le n-1; \\ 3n+1, & \text{for } i = n; \end{cases}$$

$W_3(ab) = 6n + 4;$

$W_3(bc) = 6n + 5;$

$$W_3(aa_i) = \begin{cases} 3n+i+1, & \text{for } 1 \le i \le n-1; \\ 6n, & \text{for } i = n; \end{cases}$$

$$W_3(bb_i) = \begin{cases} 4n+i, & \text{for } 1 \le i \le n-1; \\ 6n-1, & \text{for } i = n; \end{cases}$$

$$W_3(cc_i) = \begin{cases} 5n+i-1, & \text{for } 1 \le i \le n-1; \\ 6n+1, & \text{for } i = n; \end{cases}$$

By evaluating the weights of the vertex and edges, we obtain as follows:

$$\begin{aligned} W_3\left(V(S_{n,n,n})\right) &= \{W_3(a_i), W_3(b_i), W_3(c_i) | 1 \le i \le n\} \cup \{W_3(a), W_3(b), W_3(c)\} \\ &= \{2,3,\cdots,3n+1\} \cup \{6n+2, 6n+6, 6n+3\} \end{aligned}$$

$$\begin{aligned} W_3\left(E(S_{n,n,n})\right) &= W_3(aa_i), W_3(bb_i), W_3(cc_i) | 1 \le i \le n\} \cup \{W_3(ab), W(bc)\} \\ &= \{3n+2, 3n+3, \cdots, 6n+1\} \cup \{6n+4, 6n+5\} \end{aligned}$$

It can be seen that no two vertices have the same weight, no two edges have the same weight, and $W_3(V) \cap W_3(E) = \emptyset$. Thus, $f_3$ is a disjoint total irregular total labeling and $ds(S_{n,n,n}) = 3n$, for $n \ge 1$. ■

## 3. Conclusion

Based on the discussion of disjoint total irregular total labeling from double star graphs and triple star graphs, it can be concluded that their disjoint total irregularity strengths are equal to the lower bound.

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