April 2025
Volume 6 Nomor 1

p-ISSN 2723-0325 e-ISSN 2723-0333

TENSOR

Pure and Applied Mathematics Journal

PROGRAM STUDI MATEMATIKA
JURUSAN MATEMATIKA
FAKULTAS MATEMATIKA DAN ILMU PENGETAHUAN ALAM
UNIVERSITAS PATTIMURA

TENSOR
Pure and Applied Mathematics Journal

is an international academic open-access journal that gains a foothold in mathematics, and its applications are issued twice a year. The focus is to publish original research and review articles on all aspects of pure and applied Mathematics. Editorial board members of the Journal and reviewers will review submitted papers. All submitted articles should report original, previously unpublished research results, experimental or theoretical, and will be peer-reviewed. Articles submitted to the journal should meet these criteria and must not be under consideration for publication elsewhere. Manuscripts should follow the journal template and are subject to both review and editing.

Published by:

Department of Mathematics,
Faculty of Science and Technology,
Pattimura University.
Ambon
2025
Copyright© Program Studi Matematika FST UNPATTI 2025

Volume 6 Number 1 | April 2025

Person In Charge
Head of Undergraduate Program in Mathematics,
Faculty of Mathematics and Natural Sciences, Pattimura University

Editor in Chief
Dr. H. Batkunde, S.Si, M.Si

Editors
M. I. Tilukay, S.Si, M.Si (Managing and Section Editor)
L. Bakarbessy, S.Si, M.Si (Managing and Section Editor)
Z. A. Leleury, S.Si., M.Si (Copy and Production Editor)
B. P. Tomasouw, S.Si, M.Si (Copy and Production Editor)
Dr. L. K. Beay, S.Pd., M.Si (Proofreader)
N. Dahoklory (Proofreader)

Secretariat and Financial Officer
M. E. Rijoly, S.Si, M.Sc

Graphic Design
V. Y. I. Ilwaru, S.Si, M.Si

Expert Editorial Boards
Prof. Dr. Basuki Widodo, M.Sc (Institut Teknologi Sepuluh November Surabaya, Indonesia)
Prof. Dr. M. Salman A. N, M.Si (Institut Teknologi Bandung, Indonesia)
Dr. H. J. Wattimanela, S.Si., M.Si (Universitas Pattimura, Indonesia)
Dr. Al Azhary Masta, S.Si., M.Si (Universitas Pendidikan Indonesia, Indonesia)
Dr. Muh. Nur, S.Si., M.Si (Universitas Hasanudin, Indonesia)
Dr. Meta Kallista, S.Si., M.Si (Universitas Telkom,Indonesia)
Dr. Teguh Herlambang, S.Si., M.Si (Universitas Nahdlatul Ulama Surabaya, Indonesia)
Asst. Prof. Dr. Anurak Thanyacharoen (Muban Chombueng Rajabhat University, Ratchaburi, Thailand)

Publisher
Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Pattimura University, Ambon, Indonesia

Editorial Address
Program Studi Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Unversitas Pattimura
Jln. Ir. M. Putuhena, Kampus Unpatti, Poka - Ambon 97233, Provinsi Maluku, Indonesia
Contact : +62 82397854220
Email : tensormathematics@gmail.com

Volume 6 Number 1 | September 2024

TENSOR
Pure and Applied Mathematics Journal

TENSOR Pure and Applied Mathematics Journal
April | Vol. 6 | No. 1 pp. 57-66
p-ISSN: 2723-0325 e-ISSN: 2723-0333

The Rainbow Vertex Connection Number of Some Amalgamation of Two Cycles

Pranaya D. M. Taihuttu[1*], Meilin I. Tilukay[1], Francis Y. Rumlawang[1], E. M. C. Wattimena[1]
[1,] Department of Mathematics, Faculty of Science and Technology, Pattimura University, Ambon, Indonesia.
[*]Email: dhariataihuttu@gmail.com

Manuscript submitted : March 2025;
Accepted for publication : June 2025.
doi: https://doi.org/10.30598/tensorvol6iss1pp57-66

Abstract: This paper focuses on rainbow vertex coloring in a graph G, in which, for every two vertices in G, there exists a rainbow vertex path where all internal vertices have distinct colors. The rainbow vertex connection number of G, denoted by $rvc(G)$, is the minimum number of colors required to make G rainbow-vertex connected. In this paper, we determine the rainbow vertex connection number of some amalgamation of two cycles.

2010 Mathematical Subject Classification : 05C15.
Keywords: Rainbow Vertex Coloring, Rainbow Vertex Connection Number, Amalgamation graph, Cycle

1. Introduction

Graph theory is a branch of combinatorial mathematics that is widely used for problem-solving. It deals with representing discrete objects and their relationships using graphs. This field has garnered considerable attention due to its practical applications in everyday life, such as planning transportation schedules, communication networks, computer science, chemistry, circuit design, operations research (scheduling), database management. It is also used in many applications, such as coding theory [1].

In the field of graph theory research, graph labeling has recently become a topic of significant interest. Graph labeling involves assigning integers to vertices, edges, or both while adhering to specific conditions. There are three primary types of graph labeling: vertex labeling, edge labeling, and total labeling [2].

Graph coloring is a particular form of graph labeling in which colors are assigned to elements (vertices, edges, and faces) of a graph. Vertex coloring involves assigning colors to all the vertices of a graph in such a way that no two adjacent vertices share the same color. Edge coloring is a mapping that assigns a color to every edge, satisfying the condition that no two edges sharing a common vertex have the same color [3]. Face coloring of a planar graph assigns a color to each face so that no two faces sharing an edge have the same color [4].

All graphs considered in this paper are finite, simple, and undirected. We adopt the notation and terminology used by Diestel [5]. Let $G = (V(G), E(G))$ be a vertex-colored graph. A path in G is called a rainbow vertex path if its internal vertices have distinct colors. The vertex colored graph G is said to be rainbow vertex connected if there exists at least one rainbow vertex path between every pair of vertices. If a rainbow vertex coloring of G uses k colors, it is referred to as a k-rainbow vertex coloring. The rainbow vertex connection number of a connected graph G, denoted by $rvc(G)$, is the smallest positive integer k such that G has a k-rainbow vertex coloring. This concept was introduced by Krivelevich and Yuster [6].

Let G be a connected graph, let n be the order of G, and let diameter of G be denoted by $\text{diam}(G)$. Then, they provided lower and upper bounds for $rvc(G)$, namely

$$\text{diam}(G) - 1 \leq rvc(G) \leq n - 2. \tag{1}$$

Moreover, if G has c cut vertices, then

$$rvc(G) \geq c. \tag{2}$$

Furthermore, by coloring the cut vertices with distinct colors, we obtain $rvc(G) = c$. By definition, $rvc(G) = 0$ if and only if G is a complete graph, and $rvc(G) = 1$ if and only if $\text{diam}(G) = 2$.

There have been numerous intriguing findings regarding rainbow vertex connection numbers. Some of these results were presented by Li and Liu. They determined the rainbow vertex connection number of a cycle C_n for $n \geq 3$. Based on this, they proved that for any 2-connected graph G, $rvc(G) \leq rvc(C_n)$, providing a tight upper bound for the rainbow vertex connection. As a consequence, they showed that for a connected graph G with a block decomposition $B_1, B_2, \dots, B_k$ and c cut vertices, the inequality $rvc(G) \leq rvc(B_1) + rvc(B_2) + \cdots + rvc(B_k) + c$ holds [7]. Meanwhile, Simamora and Salman determined the rainbow vertex connection number of the pencil graph [8]. Furthermore, the rainbow vertex connection numbers of a simple graph class formed by graph operations are given in [9–15].

In 2016, Rosmaini [16] determined the rainbow vertex connection number of the amalgamation of several simple graphs, i.e., friendships, fans, wheels, helms, gears, and ladders. In this paper, we derive the rainbow vertex connection number of some amalgamation of two cycles. To simplify notation, we define $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

2. Main Results

Let $\{G_i \mid i \in \{1, 2, 3, \dots, t\}\}$ for $t \in \mathbb{N}$ and $t \geq 2$ be a collection of connected, finite, and simple graphs, where each G_i has a fixed vertex v_{0i}, called the central vertex. The amalgamation of G_i, denoted by $Amal(G_i, v_{0i})$, is the graph obtained by taking all the vertices and edges of G_i and identifying $v_{oi} = v_{0j}$, for all $i \neq j$. The graph $Amal(G_i, v_{0i})$ can also be written as $Amal(G_1, G_2, \dots, G_t;\ v_{01}, v_{02}, \dots, v_{0t})$. If $G_i = G_j = G$ with $v_{0i} = v_{0j}$ for each i, j, then the notation simplifies to $Amal(G, v_0, t)$.

Next, we will show $rvc(Amal(G, v_0, 2))$ for $G \cong C_n$, where C_n is a cycle of order n and $rvc(Amal(C_n, C_{2n}; v_0))$, where C_{2n} is a cycle of order $2n$.

Theorem 1.
Let n be an integer with $n \geq 3$. Let C_n be a cycle graph of order n, and let v_0 be a central vertex of C_n. Then,

$$rvc(Amal(C_n, v_0, 2)) = \begin{cases} n - 1, & \text{for } n \text{ even;} \\ n - 2, & \text{for } n \text{ odd.} \end{cases}$$

Proof.
We define the set of vertices and edges of $Amal(C_n, v_0, 2)$ as follows:
$V(Amal(C_n, v_0, 2)) = \{v_0\} \cup \{v_{i,j} | i \in [1,2], j \in [1, n-1]\}$ and
$E(Amal(C_n, v_0, 2)) = \{v_0 v_{i,j} | i \in [1,2], j \in \{1, n-1\}\} \cup \{v_{i,j} v_{i,j+1} | i \in [1,2], j \in [1, n-2]\}$. (see Figure 1)

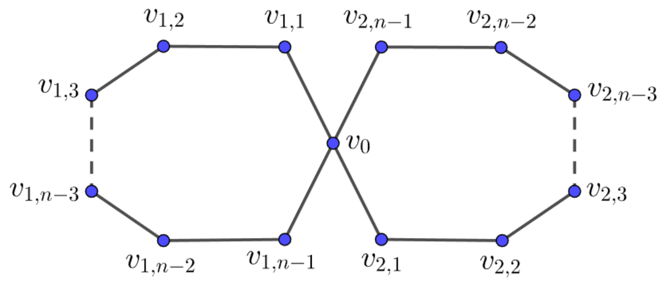

Fig. 1. $Amal(C_n, v_0, 2)$

Proof of this theorem consists of two cases.

Case 1. n even

Based on equation (1), we have

$$rvc(Amal(C_n, v_0, 2)) \geq \text{diam}(Amal(C_n, v_0, 2)) - 1 = n - 1. \quad (3)$$

In order to prove that $rvc(Amal(C_n, v_0, 2)) \leq n - 1$, we define a vertex-coloring $\alpha: V(Amal(C_n, v_0, 2)) \to [1, n-1]$ as follows:

$\alpha(v_0) = \frac{n}{2}$,

$$\alpha(v_{1,j}) = \begin{cases} j, & \text{for } j \in \left[1, \frac{n}{2}\right]; \\ j - \frac{n}{2}, & \text{for } j \in \left[\frac{n}{2} + 1, n - 1\right], \end{cases}$$

$$\alpha(v_{2,j}) = \begin{cases} j + \frac{n}{2}, & \text{for } j \in \left[1, \frac{n}{2} - 1\right]; \\ j, & \text{for } j \in \left[\frac{n}{2}, n - 1\right]. \end{cases}$$

Furthermore, we can determine that $Amal(C_n, v_0, 2)$ is rainbow-vertex connected under α. Let u and v be two vertices of $Amal(C_n, v_0, 2)$. Clearly, a rainbow $u - v$ path exists if u is adjacent to v. To establish the existence of a rainbow $u - v$ path when u and v are not adjacent, we divide the proof into six cases, as presented in Table 1.

Table 1. The Rainbow Vertex $u - v$ Path in the Graph $Amal(C_n, v_0, 2)$ for Even n

$\boldsymbol{u}$	$\boldsymbol{v}$	Condition	Rainbow-vertex path
$\boldsymbol{u}$ is adjacent to $\boldsymbol{v}$			Trivial
$\boldsymbol{v_{i,j}}$	$v_{i,k}$	$i \in [1,2],\ j,k \in [1, n-1]$	
		$j < k,\ k - j \leq \frac{n}{2}$	$v_{i,j}, v_{i,j+1}, v_{i,j+2}, \cdots, v_{i,k}$
		$j < k,\ k - j > \frac{n}{2}$	$v_{i,j}, v_{i,j-1}, v_{i,j-2}, \cdots, v_0, v_{i,n-1}, v_{i,n-2}, \cdots, v_{i,k}$
$\boldsymbol{v_{1,j}}$	$v_{2,k}$	$j,k \in \left[1, \frac{n}{2}\right]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j \in \left[1, \frac{n}{2}\right],\ k \in \left[\frac{n}{2} + 1, n - 1\right]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,n-1}, v_{2,n-2}, \cdots, v_{2,k}$
		$j \in \left[\frac{n}{2} + 1, n - 1\right],\ k \in \left[1, \frac{n}{2}\right]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j,k \in \left[\frac{n}{2} + 1, n - 1\right]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,n-1}, v_{2,n-2}, \cdots, v_{2,k}$

For other cases, there is a rainbow vertex path that is a subpath of one of the rainbow vertex paths in the cases above. Hence, we conclude that

$$rvc(Amal(C_n, v_0, 2)) \leq n - 1. \quad (4)$$

From equation (3) and (4), we have $rvc(Amal(C_n, v_0, 2)) = n - 1$.

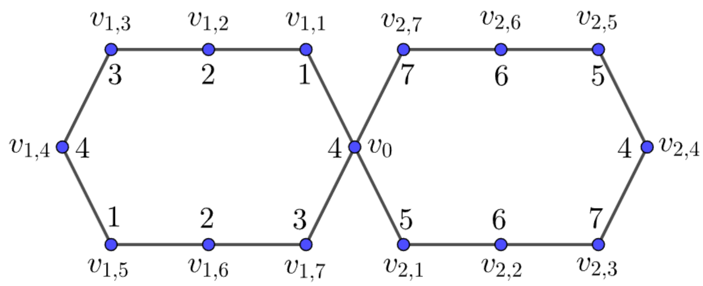

Fig. 2. Rainbow vertex coloring of $Amal(C_8, v_0, 2)$

Case 2. n odd
Based on equation (1), we have

$$rvc\big(Amal(C_n, v_0, 2)\big) \geq \text{diam}\big(Amal(C_n, v_0, 2)\big) - 1 = (n-1) - 1 = n - 2. \tag{5}$$

In order to prove that $rvc\big(Amal(C_n, v_0, 2)\big) \leq n - 2$, we define a vertex-coloring $\alpha : V\big(Amal(C_n, v_0, 2)\big) \rightarrow [1, n-2]$ as follows

$$\alpha(v_0) = \left\lfloor \frac{n}{2} \right\rfloor ,$$

$$\alpha(v_{1,j}) = \begin{cases} j, & \text{for } j \in \left[1, \left\lceil \frac{n}{2} \right\rceil\right]; \\ j - \left\lceil \frac{n}{2} \right\rceil, & \text{for } j \in \left[\left\lceil \frac{n}{2} \right\rceil + 1, n-1\right], \end{cases}$$

$$\alpha(v_{2,j}) = \begin{cases} j + \left\lfloor \frac{n}{2} \right\rfloor, & \text{for } j \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right]; \\ j - 1, & \text{for } j \in \left[\left\lfloor \frac{n}{2} \right\rfloor, n-1\right]. \end{cases}$$

Furthermore, we can determine that $Amal(C_n, v_0, 2)$ is rainbow vertex connected under α. Let u and v be two vertices of $Amal(C_n, v_0, 2)$. Clearly, a rainbow $u - v$ path exists if u is adjacent to v. To establish the existence of a rainbow $u - v$ path when u and v are not adjacent, we devide the proof into six cases as presented in Table 2.

Table 2. The Rainbow Vertex $u - v$ Path in the Graph $Amal(C_n, v_0, 2)$ for Odd n

$\boldsymbol{u}$	$\boldsymbol{v}$	Condition	Rainbow-vertex path
$\boldsymbol{u}$ is adjacent to $\boldsymbol{v}$			Trivial
$\boldsymbol{v_{i,j}}$	$v_{i,k}$	$i \in [1,2],\ j,k \in [1, n-1]$	
		$j < k,\ k - j \leq \left\lceil \frac{n}{2} \right\rceil$	$v_{i,j}, v_{i,j+1}, v_{i,j+2}, \cdots, v_{i,k}$
		$j < k,\ k - j > \left\lceil \frac{n}{2} \right\rceil$	$v_{i,j}, v_{i,j-1}, v_{i,j-2}, \cdots, v_0, v_{i,n-1}, v_{i,n-2}, \cdots, v_{i,k}$
$\boldsymbol{v_{1,j}}$	$v_{2,k}$	$j,k \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor\right]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor\right],\ k \in \left[\left\lceil \frac{n}{2} \right\rceil, n-1\right]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,n-1}, v_{2,n-2}, \cdots, v_{2,k}$
		$j \in \left[\left\lceil \frac{n}{2} \right\rceil, n-1\right],\ k \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor\right]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j,k \in \left[\left\lceil \frac{n}{2} \right\rceil, n-1\right]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,n-1}, v_{2,n-2}, \cdots, v_{2,k}$

For other cases, there is a rainbow vertex path that is a sub-path of one of the rainbow vertex paths in the cases above. Hence, we conclude that

$$rvc\big(Amal(C_n, v_0, 2)\big) \leq n - 2. \tag{6}$$

From equation (5) and (6), we have $rvc\big(Amal(C_n, v_0, 2)\big) = n -$

2. ∎

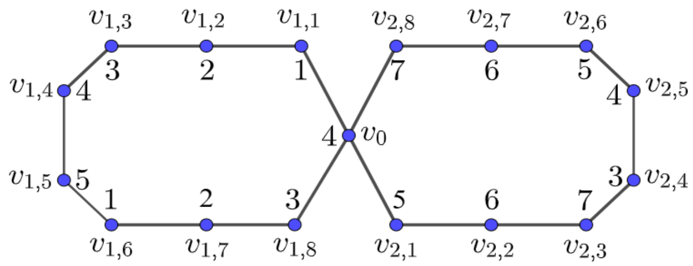

Fig. 3. Rainbow vertex coloring of $Amal(C_9, v_0, 2)$

Theorem 2.
Let n be an integer with $n \geq 3$. Let C_n be a cycle graph of order n, and let v_0 be a central vertex of C_n. Then,

$$rvc\big(Amal(C_n, C_{2n}; v_0)\big) = \begin{cases} \frac{3}{2}n - 1, & \text{for } n \text{ even;} \\ \frac{3}{2}n - \frac{3}{2}, & \text{for } n \text{ odd.} \end{cases}$$

Proof.
We define the set of vertices and edges of $Amal(C_n, C_{2n}; v_0)$ as follows:
$V\big(Amal(C_n, C_{2n}; v_0)\big) = \{v_0\} \cup \{v_{1,j} \mid j \in [1, n-1]\} \cup \{v_{2,j} \mid j \in [1, 2n-1]\}$ and
$E\big(Amal(C_n, C_{2n}; v_0)\big) = \{v_0 v_{1,j} \mid j \in \{1, n-1\}\} \cup \{v_0 v_{2,j} \mid j \in \{1, 2n-1\}\} \cup \{v_{1,j} v_{1,j+1} \mid j \in [1, n-2]\} \cup \{v_{2,j} v_{2,j+1} \mid j \in [1, 2n-2]\}$. (see Figure 4)

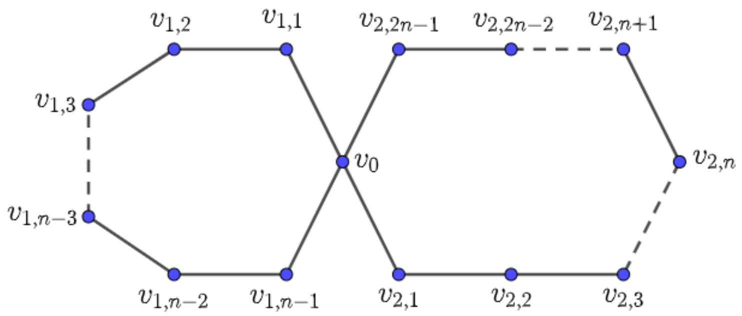

Fig. 4. $Amal(C_n, C_{2n}; v_0)$

Proof of this theorem consists of two cases.
Case 1. n even
Based on equation (1), we have

$$rvc\big(Amal(C_n, C_{2n}; v_0)\big) \geq \text{diam}\big(Amal(C_n, C_{2n}; v_0)\big) - 1 = \tfrac{3}{2}n - 1. \quad (7)$$

In order to prove that $rvc\big(Amal(C_n, C_{2n}; v_0)\big) \leq \frac{3}{2}n - 1$, we define a vertex-coloring $\beta: V\big(Amal(C_n, C_{2n}; v_0)\big) \to \left[1, \frac{3}{2}n - 1\right]$ as follows:

$\beta(v_0) = \frac{n}{2}$,

$$\beta(v_{1,j}) = \begin{cases} j, & \text{for } j \in \left[1, \frac{n}{2}\right]; \\ j - \frac{n}{2}, & \text{for } j \in \left[\frac{n}{2}+1, n-1\right], \end{cases}$$

$$\beta(v_{2,j}) = \begin{cases} j + \frac{n}{2}, & \text{for } j \in [1, n-1]; \\ j - \frac{n}{2}, & \text{for } j \in [n, 2n-1]. \end{cases}$$

Furthermore, we can determine that $Amal(C_n, C_{2n}; v_0)$ is rainbow-vertex connected under β. Let u and v be two vertices of $Amal(C_n, C_{2n}; v_0)$. Clearly, a rainbow $u - v$ path exists if u is adjacent to v. To establish the existence of a rainbow $u - v$ path when u and v are not adjacent, we divide the proof into eight cases, as presented in Table 3.

Table 3. The Rainbow Vertex $u - v$ Path in the Graph $Amal(C_n, C_{2n}; v_0)$ for Even n

$\boldsymbol{u}$	$\boldsymbol{v}$	Condition	Rainbow-vertex path
$\boldsymbol{u}$ is adjacent to $\boldsymbol{v}$			Trivial
$\boldsymbol{v_{i,j}}$	$v_{i,k}$	$i = 1,\ j, k \in [1, n-1]$	
		$j < k,\ k - j \le \frac{n}{2}$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,k}$
		$j < k,\ k - j > \frac{n}{2}$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_0, v_{1,n-1}, v_{1,n-2}, \cdots, v_{1,k}$
		$i = 2,\ j, k \in [1, 2n-1]$	
		$j < k,\ k - j \le n$	$v_{2,j}, v_{2,j+1}, v_{2,j+2}, \cdots, v_{2,k}$
		$j < k,\ k - j > n$	$v_{2,j}, v_{2,j-1}, v_{2,j-2}, \cdots, v_0, v_{2,2n-1}, v_{2,2n-2}, \cdots, v_{2,k}$
$\boldsymbol{v_{1,j}}$	$v_{2,k}$	$j \in \left[1, \frac{n}{2}\right],\ k \in [1, n]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j \in \left[1, \frac{n}{2}\right]$, $k \in [n+1, 2n-1]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,2n-1}, v_{2,n-2}, \cdots, v_{2,k}$
		$j \in \left[\frac{n}{2}+1, n-1\right],\ k \in [1, n]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j \in \left[\frac{n}{2}+1, n-1\right]$, $k \in [n+1, 2n-1]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,2n-1}, v_{2,2n-2}, \cdots, v_{2,k}$

For other cases, there is a rainbow vertex path that is a sub-path of one of the rainbow vertex paths in the cases above. Hence, we conclude that

$$rvc\big(Amal(C_n, C_{2n}; v_0)\big) \le \frac{3}{2}n - 1. \tag{8}$$

From equation (7) and (8), we have $rvc\big(Amal(C_n, C_{2n}; v_0)\big) = \frac{3}{2}n - 1$.

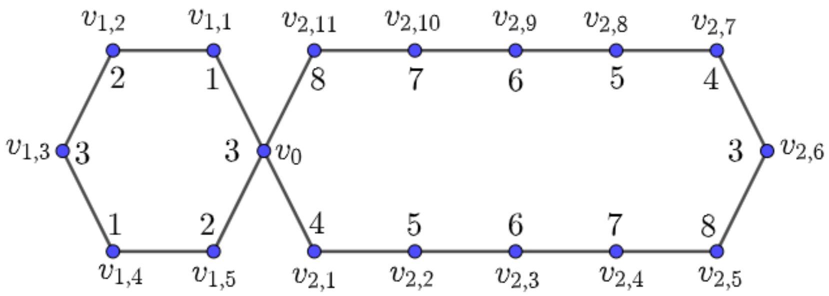

Fig. 5. Rainbow vertex coloring of $Amal(C_6, C_{12}: v_0)$

Case 2. n odd

Based on equation (1), we have

$$rvc\big(Amal(C_n, C_{2n}; v_0)\big) \geq \operatorname{diam}\big(Amal(C_n, C_{2n}; v_0)\big) - 1 = \left(\tfrac{3}{2}n - \tfrac{1}{2}\right) - 1 = \tfrac{3}{2}n - \tfrac{3}{2}. \qquad (9)$$

In order to prove that $rvc\big(Amal(C_n, C_{2n}; v_0)\big) \leq \frac{3}{2}n - \frac{3}{2}$, we define a vertex-coloring $\beta: V\big(Amal(C_n, C_{2n}; v_0)\big) \to \left[1, \frac{3}{2}n - \frac{3}{2}\right]$ as follows:

$\beta(v_0) = \left\lfloor \frac{n}{2} \right\rfloor$,

$$\beta(v_{1,j}) = \begin{cases} j, & \text{for } j \in \left[1, \left\lceil \frac{n}{2} \right\rceil\right]; \\ j - \left\lceil \frac{n}{2} \right\rceil, & \text{for } j \in \left[\left\lceil \frac{n}{2} \right\rceil + 1, n-1\right], \end{cases}$$

$$\beta(v_{2,j}) = \begin{cases} j + \left\lfloor \frac{n}{2} \right\rfloor, & \text{for } j \in [1, n-1]; \\ j - \left\lceil \frac{n}{2} \right\rceil, & \text{for } j \in [n, 2n-1]. \end{cases}$$

Furthermore, we can determine that $Amal(C_n, C_{2n}; v_0)$ is rainbow vertex connected under β. Let u and v be two vertices of $Amal(C_n, C_{2n}; v_0)$. Clearly, a rainbow $u-v$ path exists if u is adjacent to v. To establish the existence of a rainbow $u-v$ path when u and v are not adjacent, we devide the proof into eight cases as presented in Table 4.

Table 4. The Rainbow Vertex $u-v$ Path in the Graph $Amal(C_n, C_{2n}; v_0)$ for Odd n

$\boldsymbol{u}$	$\boldsymbol{v}$	Condition	Rainbow-vertex path
$\boldsymbol{u}$ is adjacent to $\boldsymbol{v}$			Trivial
$\boldsymbol{v_{i,j}}$	$v_{i,k}$	$i = 1$, $j,k \in [1, n-1]$	
		$j < k,\ k - j \leq \left\lceil \frac{n}{2} \right\rceil$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,k}$
		$j < k,\ k - j > \left\lceil \frac{n}{2} \right\rceil$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_0, v_{1,n-1}, v_{1,n-2}, \cdots, v_{1,k}$
		$i = 2,\ j,k \in [1, 2n-1]$	
		$j < k,\ k - j \leq n$	$v_{2,j}, v_{2,j+1}, v_{2,j+2}, \cdots, v_{2,k}$
		$j < k,\ k - j > n$	$v_{2,j}, v_{2,j-1}, v_{2,j-2}, \cdots, v_0, v_{2,2n-1}, v_{2,2n-2}, \cdots, v_{2,k}$
$\boldsymbol{v_{1,j}}$	$v_{2,k}$	$j \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor\right],\ k \in [1, n]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor\right],\ k \in [n+1, 2n-1]$	$v_{1,j}, v_{1,j-1}, v_{1,j-2}, \cdots, v_{1,1}, v_0, v_{2,2n-1}, v_{2,2n-2}, \cdots, v_{2,k}$
		$j \in \left[\left\lceil \frac{n}{2} \right\rceil, n-1\right],\ k \in [1, n]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,1}, v_{2,2}, \cdots, v_{2,k}$
		$j \left[\left\lceil \frac{n}{2} \right\rceil, n-1\right],$ $k \in [n+1, 2n-1]$	$v_{1,j}, v_{1,j+1}, v_{1,j+2}, \cdots, v_{1,n-1}, v_0, v_{2,2n-1}, v_{2,2n-2}, \cdots, v_{2,k}$

For other cases, there is a rainbow vertex path that is a sub-path of one of the rainbow vertex paths in the cases above. Hence, we conclude that

$$rvc\big(Amal(C_n, C_{2n}; v_0)\big) \leq \tfrac{3}{2}n - \tfrac{3}{2}. \qquad (10)$$

From equation (9) and (10), we have $rvc\big(Amal(C_n, v_0, 2)\big) = \frac{3}{2}n - \frac{3}{2}$. ■

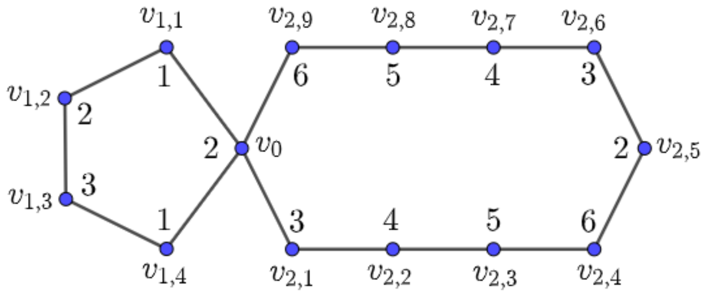

Fig. 6. Rainbow vertex coloring of $Amal(C_5, C_{10}: v_0)$

3. Conclusion

Based on the results and discussion, the rainbow vertex connection number of the amalgamation of two cycles is determined as follows:
Let n be an integer with $n \geq 3$. Let C_n be a cycle graph of order n, and let v_0 be a central vertex of C_n. Then,

$$rvc\big(Amal(C_n, v_0, 2)\big) = \begin{cases} n-1, & \text{for } n \text{ even;} \\ n-2, & \text{for } n \text{ odd,} \end{cases}$$

and

$$rvc\big(Amal(C_n, C_{2n}; v_0)\big) = \begin{cases} \frac{3}{2}n - 1, & \text{for } n \text{ even;} \\ \frac{3}{2}n - \frac{3}{2}, & \text{for } n \text{ odd.} \end{cases}$$

Next, we will determine the rainbow vertex connection number of the amalgamation of two cycles with for any different orders.

References

[1] Prathik, A., Uma, K., & Anuradha, J. (2016). An Overview of Application of Graph Theory. *International Journal of ChemTech Research Vol. 9*, No. 02, pp. 242 – 248.

[2] Galian, J. A. (2022). A dynamic survey of graph labeling. Electronic Journal of Combinatorics. 25, 1 623 Retrieved December 2^{nd}, 2020, from: https://www. combinatorics.org/files/Surveys/ds6/ds6v25 2022. pdf.

[3] Formanowicz, P., & Tanas, K. (2012). A Survey of Graph Coloring - Its Types, Methods and Applications. *Foundation of Computing and Decision Sciences Vol. 37*, No. 3, pp. 223 – 238.

[4] Nisviasari, R., Dafik, Maryati, T. K., Agustin, I. H., & Kurniawati, E. Y. (2019). Local super antimagic total face coloring of planar graphs. *IOP Conference Series: Earth and Environmental Science*, 243 (1), 012117.

[5] Diestel, R. (2017). Graph Theory, 5^{th}*edition.* Springer.

[6] Krivelevich, M., & Yuster, R. (2010). The rainbow connection of a graph is (at most) reciprocal to its minimum degree. *J. Graph Theory*, 63 (3), 185 – 191.

[7] Li, X., & Liu, S. (2011). Rainbow vertex-connection number of 2-connected graphs.

[8] Simamora, D. N. S., & Salman A. N. M. (2015). The rainbow (vertex) connection number of pencil graphs. *Procedia Computer Science*, 74, 138 – 142.

[9] Bustan A. W. (2016). Bilangan Terhubung Titik Pelangi Untuk Graf Lingkaran Bintang $(S_m C_n)$. *Barekeng: Jurnal Ilmu Matematika dan Terapan*, 10 (2), 77 – 81.

[10] Agustina M., Dafik, Slamin, Kusbudiono. (2017). On the Rainbow Vertex Connection Number of Edge Comb of Some Graph. *Proceeding The* 1^{st} *IBSC: Towards The Extended Use Of Basic Science For*

Enhancing Health, Environment, Energy And Biotechnology (pp. 340 – 342).

[11] Bustan A. W., & Salman A. N. M. (2018). The Rainbow Vertex-Connection Number of Star Fan Graphs. *CAUCHY: Jurnal Matematika Murni dan Aplikasi*, 5 (3), 112 – 116.

[12] Bustan A. W., & Salman A. N. M. (2019). The Rainbow Vertex Connection Number of Star Wheel Graphs. *AIP Conference Proceedings: Vol. 2202*.

[13] Fauziah D. A., Dafik, Agustin I. H., & Alfarisi R. (2019). The Rainbow Vertex Connection Number of Edge Corona Product Graphs. *IOP Conf. Series: Earth and Environmental Science*, 243, 12 – 20.

[14] Lihawa I., Ismail S., Hasan I. K., Yahya L., Nasib S. K., & Yahya N. I. (2022). Bilangan Terhubung Titik Pelangi pada Graf Hasil Operasi Korona Graf Prisma $(P_{m,2})$ dan Graf Lintasan (P_3). *Jambura J. Math. Vol. 4, No. 1*, pp. 145 – 151.

[15] Yahya N. I., Fatmawati A., Nurwan, & Nasib S. K. (2023). Rainbow Vertex-Connection Number on Comb Product Operation of Cycle Graph (C_4) and complete Bipartite Graph $(K_{3,n})$.

[16] Rosmaini. (2016). Bilangan Terhubung Titik Pelangi Untuk Beberapa Graf Amalgamasi. Tesis, Institut Teknologi Bandung, Bandung.