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Fixed Point Theorem in 2-Normed Spaces

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Abstract: In this paper we prove a fixed point theorem in a complete 2-normed Spaces. We define a norm derived from 2-norm. To get the theorem proved we first study some convergent and Cauchy sequences, and contractive mappings in 2-normed spaces.

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1. Introduction

A normed space is a vector space equipped with a function called norm. Geometrically, a norm is a tool to measure length of a vector.

Definition 1. [5] Let X be a vector space with $\dim(X) \geq 2$. A mapping $||\cdot||: X \to \mathbb{R}$ that satisfies

(1). $\|x\| \geq 0$, for all $x \in X$;
$\|x\| = 0$ if and only if $x = 0$,

(2). $\|\alpha\, x\| = |\alpha|\|x\|$; for all $\alpha \in \mathbb{R}$ and $x \in X$,

(3). $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$

Is called a **norm**. A pair of $(X, \|\cdot,\cdot\|)$ is called a **normed space**.

In 1960's Gahler introduced a concept of n-normed spaces as a generalization of a concept of normed spaces. This space is equipped by an n-norm. The n-normed is used to measure volume of a parallelepiped spaned by n vectors. Especially for $n = 2$, the 2-norm is a tool to measure an area spaned by 2 vectors. Te concept of 2-normed space was studied further by many researchers, for instance see [1,4,6]. Now, we present some basic definition and properties of 2-normed spaces.

Definition 2. [2] Let X be a vector space with $\dim(X) \geq 2$. A mapping $||\cdot,\cdot||: X \to \mathbb{R}$ that satisfies

(N1). $\|x, y\| \geq 0$, for all $x, y \in X$;
$\|x, y\| = 0$ if and only if x, y linearly dependent,

(N2). $\|x, y\| = \|y, x\|$; for all $x, y \in X$,

(N3). $\|\alpha\, x, y\| = |\alpha|\|x, y\|$; for all $\alpha \in \mathbb{R}$ and $x, y \in X$,

(N4). $\|x + z, y\| \leq \|x, y\| + \|z, y\|$, for all $x, y, z \in X$

Is called a **2-norm**. A pair of $(X, \|\cdot,\cdot\|)$ is called a **2-normed space**.

Note that in 2-normed space$(X, \|\cdot,\cdot\|)$ we have

$$\|x_1, x_2\| = \|x_1, x_2 + \alpha x_1\|,$$

For all $\alpha \in \mathbb{R}$ and $x_1, x_2 \in X$.

Definition 3. [2] A sequence $\{x_k\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be **convergent** if there is an $x \in X$ such that $\lim_{k\to\infty} \|x_k - x, z\| = 0$ for all $z \in X$.
If $\{x_k\}$ converges to x the we denote it by $x_k \to x$ as $k \to \infty$. The point x is called limit point of x_k.

Definition 4. [2] A sequence $\{x_k\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be a **Cauchy sequence** if there is an $x \in X$ such that $\lim_{k,l\to\infty} \|x_k - x_l, z\| = 0$ for all $z \in X$.

Lemma 5. If A sequence $\{x_k\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is convergent, then $\{x_k\}$ is a Cauchy sequence.

Definition 6. A 2-normed space is called **complete** if every Cauchy sequence is convergent.

Moreover, the complete 2-normed space is called a **2-Banach space.**

Definition 7. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A set $K \subset X$ is said to be **closed** if the limit point of every convergent sequence in K is also in K.

1. Main Results

In this section, we define a normed derived from 2-norm and use this norm to prove a fixed point theorem in 2-normed space. We begin with defining the norm.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $Y = \{y_1, y_2\}$ be a linearly independent set in X, we define a function in X by

$$\|x\| = \|x, y_1\| + \|x, y_2\| \tag{1}$$

One can see that the function $\|\cdot\|: X \to \mathbb{R}$ defines in (1) defines a norm in X.

Theorem 8. $(X, \|\cdot\|)$ is a norm space, with $\|\cdot\|$ is a norm defined in (1).

Proof. We just need to prove that normed defined in (1) as a norm in X.

(1). By using (N1), one can see that for every $x \in X$ we have

$$\|x\| = \|x, y_1\| + \|x, y_2\| \geq 0,$$

because each term on the above equation will greater or equals 0.
If $x = 0$, from then (N1) we have $\|x, y_1\| = 0$ and $\|x, y_2\| = 0$, which means $\|x\| = 0$.
If $\|x\| = 0$ then $\|x, y_1\| + \|x, y_2\| = 0$. Because each term is nonnegative then we shoud have $\|x, y_1\| = 0$ and $\|x, y_2\| = 0$. This means x is a vector that dependent only to y_1 and also dependent only to y_2. The vector x must be 0.

(2). For any $x \in X$ and $\alpha \in \mathbb{R}$, $\|\alpha\, x\| = \|\alpha\, x, y_1\| + \|\alpha\, x, y_2\|$.
By using (N3) we have $\|\alpha\, x, y_1\| + \|\alpha\, x, y_2\| = |\alpha|\, (\|x, y_1\| + \|x, y_2\|) = |\alpha|\, \|x\|$. Then we have $\|\alpha\, x\| = |\alpha|\|x\|$; for all $\alpha \in \mathbb{R}$ and $x \in X$.

(3). For any $x, y \in X$ we have $\|x + y\| = \|x + y, y_1\| + \|x + y, y_2\|$. By using (N4) we also have

$$\|x + y, y_1\| + \|x + y, y_2\| \leq \|x, y_1\| + \|y, y_1\| + \|x, y_2\| + \|y, y_2\|.$$

This means $\|x + y, y_1\| + \|x + y, y_2\| \leq \|x, y_1\| + \|x, y_2\| + \|y, y_1\| + \|y, y_2\| = \|x\| + \|y\|$. Hence,

$$\|x + y\| = \|x\| + \|y\|.$$

We proved that the norm defined in (1) is a norm as desired then A pair of $(X, \|\cdot\|)$ is a normed space. □

For simplicity, from now on we call the norm defined in (1) 'derived norm'. We will using this norm to prove a fixed point theorem in 2-normed space. Before that, we show in this following proposition a convergent sequence with respect to 2-norm also convergent with respect to derived norm.

Proposition 9. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $Y = \{y_1, y_2\}$ be a linearly independent set. If x_k converges to an $x \in X$ in the 2-norm, then x_k also converges to x in derived norm.
Proof. If x_k converges to an $x \in X$ in the 2-norm then $\lim_{k\to\infty}\|x_k - x, z\| = 0$ for all $z \in X$. We can write $\lim_{k\to\infty}\|x_k - x, y_i\| = 0$ for $i = 1,2$. Hence $\lim_{k\to\infty}\|x_k - x\| = \lim_{k\to\infty}(\|x_k - x, y_1\| + \|x_k - x, y_2\|) = 0$.

Recall the standard case for 2-normed space. Let X be a real inner product space with $\dim(X) \geq 2$. We equipped X with standard 2-norm

$$\|x_1, x_2\|_s := \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{vmatrix}^{\frac{1}{2}} = (\|x_1\|_X\|x_1, x_2\|_X - \langle x_1, x_2\rangle^2)^{\frac{1}{2}}$$

With $\langle\cdot,\cdot\rangle$ denotes an inner product in X. One can see that the norm $\|\cdot\|_X$ is an induced norm, where $\|x\|_X = \langle x, x\rangle^{\frac{1}{2}}$ and $\|x_1, x_2\|_s$ is the area spanned by x_1 and x_2.

Moreover, lete_1, e_2 be two orthonormal vectors, then derived norm in (1) can be rewritten as

$$\|x\|_d = \|x, e_1\| + \|x, e_2\|. \tag{2}$$

Next, we have this following proposition. □

Proposition 10. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $Y = \{y_1, y_2\}$ be a linearly independent set. If x_k converges to an $x \in X$ in the 2-norm, then x_k also converges to x in derived norm.
Proof. The proof is similar with proof of Proposition 9.

Corollary 11. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $Y = \{y_1, y_2\}$ be a linearly independent set. If X is complete with respect to $\|\cdot,\cdot\|$, then X is also complete with respect to derived norm.

Proposition 12. Standard 2-norm is equivalent with derived norm defined on (2). Pricesely, we have

$$\frac{1}{2}\|x\|_d \leq \|x\|_X \leq \sqrt{2}\,\|x\|_d.$$

Proof. Let $x \in X$ For $i = 1,2$, we write $e_i = e_i' + e_i^{\perp}$, with $e_i' \in \text{span}\{x\}$ and $e_i^{\perp} \perp \text{span}\{x\}$. Then for $i = 1,2$, we have

$$\begin{aligned} \|x, e_i\| &= \|x, e_i^{\perp}\| \\ &= \begin{vmatrix} \langle x, x\rangle & 0 \\ 0 & \langle e_1^{\perp}, e_1^{\perp}\rangle \end{vmatrix}^{\frac{1}{2}} \\ &\leq \|x\|_X \end{aligned}$$

Hence we have $\frac{1}{2}\|x\|_d \leq \|x\|_X$.

Further, take a unit vector $e = \alpha_1 e_1 + \alpha_2 e_2$ such that $e \in \text{span}\{x\}$. By using (N3) and (N4) we have

$$\begin{aligned} \|x\|_X &= \|x, e\| \\ &\leq |\alpha_1|\|x, e_1\| + |\alpha_2|\|x, e_2\| \\ &\leq (|\alpha_1| + |\alpha_2|)\|x\|_d \end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$|\alpha_1| + |\alpha_2| \leq (1 + 1)^{\frac{1}{2}}(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}} = \sqrt{n}.$$

Hence we obtain

$$\|x\|_X \leq \sqrt{2}\, \|x\|_d.$$

This completes the proof. □

Next, we define a closed set in 2-normed space with respect to derived norm.

Definition 13. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $Y = \{y_1, y_2\}$ be a linearly independent set. A set $K \subset X$ is said to be **bounded** if there is an $M > 0$ such that for all $x \in K$ we have

$$\|x\|_d \leq M.$$

Recall that Harikrishnan and Ravindran [3] defined contractive mappings in 2-normed space as following

Definition 14. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. Then the mapping $f: X \to X$ is said to be a **contractive mapping** if there exist a $C \in (0,1)$ such that

$$\|f(x) - f(y), z\| \leq C\|x - y, z\|,$$

For all $x, y, x \in X$.

Now we define contractive mappings with respect to derived norm in 2-normed space as following

Definition 15. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, $Y = \{y_1, y_2\}$ be a linearly independent set, and $\|\cdot\|$ be a derived norm. Then the mapping $f: X \to X$ is said to be a **contractive mapping with respect to derived norm** if there exist a $C \in (0,1)$ such that

$$\|f(x) - f(y)\| \quad \leq C\|x - y, z\| \quad ,$$

For all $x, y, x \in X$.

We will prove a fixed point theorem of a contractive mapping in a complete 2-normed spaces with respect to the derived norm. To prove this theorem, we recall Picard Iteration. For any $x_0 \in X$, the sequence $\{x_k\}$ in X given by

$$x_k = f(x_{k-1}) = f^k(x_0), k = 1,2, \dots$$

is called a sequence of successive approximation with initial value x_0. Next is a fixed point theorem ini 2-normed space.

Theorem 16. Let Let $(X, \|\cdot,\cdot\|)$ be a complete 2-normed space, $K \subset X$ be a closed and bounded set, $Y = \{y_1, y_2\}$ be a linearly independent set, and $\|\cdot\|$ be a derived norm. If $f: K \to K$ be a contractive mapping then K has a unique fixed point.

Proof. Let $x_0 \in K$ and $\{x_k\}$ is a sequence in K such that

$$x_k = f(x_{k-1}) = f^k(x_0), n = 1,2, \dots$$

For $x_0, x_1 \in K$ we have

$$\|f^2(x_0) - f^2(x_1)\| = \left\|f\big(f(x_0)\big) - f(f(x_1))\right\|,$$

f contractive, so there is a $C \in (0,1)$ such that

$$\|f^2(x_0) - f^2(x_1)\| = C\|f(x_0) - f(x_1)\|.$$

Again, because f is a contractive mapping then we have

$$\|f^2(x_0) - f^2(x_1)\| = C^2\|x_0 - x_1\|.$$

By using Induction we obtain

$$\left\|f^k(x_0) - f^k(x_1)\right\| = C^k\|x_0 - x_1\|.$$

Next, we show that the sequence x_k is a Cauchy sequence. Let $k, l \in \mathbb{N}$ without loss of generality we assume that $l > k$ and $l = k + p$, with $p \in \mathbb{N}$. By using triangle inequality, we have

$$\|x_k - x_l\| = \left\|x_k - x_{k+p}\ \right\| \leq \|x_k - x_{k+1}\| + \cdots + \left\|x_{k+p-1} - x_{k+p}\right\|.$$

Using the above properties of sequence $\{x_k\}$and fact that f is a contractive mappings, we obtain

$$\|x_k - x_l\| \le \|f^k(x_0) - f^k(x_1)\| + \cdots + \|f^{k+p-1}(x_0) - f^{k+p-1}(x_1)\|$$
$$\le (C^k + \cdots + C^{k+p-1})\|x_0 - x_1\|$$

Since K is bounded and $x_0, x_1 \in K$, then there is an $M > 0$ such that $\|x_0 - x_1\| \le M$. We have

$$\|x_k - x_l\| \le (C^k + \cdots + C^{k+p-1})M,$$

Or we can write

$$\|x_k - x_l\| \le (C^k + \cdots + C^{l-1})M.$$

Because $C \in (0,1)$ then

$$\lim_{k,l\to\infty} \|x_k - x_l\| \le \lim_{k,l\to\infty} (C^k + \cdots + C^{l-1})M = 0.$$

This means $\{x_k\}$ is a Cauchy sequence. Moreover, since $(X, \|\cdot,\cdot\|)$ is a complete space then from Corollary 11, we have $(X, \|\cdot\|)$ is also a complete space. Consequently $\{x_k\}$ is a convergent sequence. Let $x_k \to x$, since $\{x_k\} \subset K$and K is a closed set, then $x \in K$. Furthermore, f is a contractive mapping, by using properties of sequence $\{x_k\}$ we have

$$f(x) = \lim_{x\to\infty} f(x_k) = \lim_{x\to\infty} x_{k+1} = x.$$

Therefore, f has one fixed point in K. Next we show that the fixed point is unique. Assume that there is another fixed point of f in K, namely x'. The mapping f is a contractive mapping so there is a $C \in (0,1)$ such that

$$\|x - x'\| = \|f(x) - f(x')\| \le C\|x - x'\|.$$

This condition is satisfied if and only if $\|x - x'\| = 0$ (If $\|x - x'\| \ne 0$, then $C = 0$ or $C = 1$). We obtain, $x = x'$, which means the fixed point is unique. □

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