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Proper Inclusion Between Vanishing Morrey Spaces and Morrey Spaces

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Abstract: In this paper, I give an explicit function which belongs to the Morrey spaces but not in the vanishing Morrey spaces. Therefore, I obtain that the Morrey spaces contain the vanishing Morrey spaces properly.

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1. Introduction and Statement of The Main Result

Let $1 \leq p < \infty$ and $0 < \lambda < n$. A function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to be an element of the **Morrey spaces** $L^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} \left(r^{-\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

For more information regarding to the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, see [1]. Now, for every $f \in L^{p,\lambda}(\mathbb{R}^n)$, we set

$$\mathcal{M}_f(r) = \sup_{x\in\mathbb{R}^n} \left(r^{-\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

The function f belongs to the **vanishing Morrey spaces** $VL^{p,\lambda}(\mathbb{R}^n)$ if

$$\lim_{r\to 0} \mathcal{M}_f(r) = 0.$$

The vanishing Morrey spaces were introduced in [2] and have some applications in elliptic partial differential equations, operator theory, and approximation in Morrey spaces [3, 4, 5].

It is clear from the definitions above that $VL^{p,\lambda}(\mathbb{R}^n)$ is a subset of $L^{p,\lambda}(\mathbb{R}^n)$. In [4], it stated that this inclusion is proper without giving an explicit function which belongs to $L^{p,\lambda}(\mathbb{R}^n)$ but not in $VL^{p,\lambda}(\mathbb{R}^n)$. In this paper, by using the idea of the proper inclusion between the bounded Stummel modulus classes and the Stummel classes [6] (see also to be published [7] and [8]), we will give an example of that function. Although there are some inclusion relations between the Morrey spaces and the Stummel classes for some appropriate parameters [6, 9], but in general these relations may fail (submitted for publication [10]).

Our main result is stated in the following theorem.

Theorem 1. *The vanishing Morrey spaces contain the Morrey spaces properly.*

The proof of this theorem will be given in the next section.

2. Proof Of The Main Result

To prove the Theorem 1, we need to give an explicit function belongs to the Morrey spaces but not in the vanishing Morrey spaces. The positive constant $C = C(n, \lambda)$, which means depending only on n and λ, appears in this paper may be vary from line to line.

Let n be an integer number such that $n \geq 2$, $0 < \lambda < n$, and $1 \leq p < \infty$. For every integer $k \geq 3$, setting $x_k = (2^{-k}, 0, \ldots, 0) \in \mathbb{R}^n$ and

$$f_k(y) = \begin{cases} 8^{(n-\lambda)k} : y \in B(x_k, 8^{-k}) \\ 0 \qquad : y \notin B(x_k, 8^{-k}). \end{cases}$$

We define a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ which its formula is given by

$$f(y) = \left(\sum_{k=3}^{\infty} f_k(y) \right)^{\frac{1}{p}}. \tag{1}$$

We claim that this function is an element of the Morrey spaces and is not an element of the vanishing Morrey spaces. We need the following lemmas to prove this claim. In the rest of this paper, the notation of the function f defined in (1).

Lemma 1. $f \in L^p_{loc}(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$. Note that

$$\int_{B(x,r)} |f(y)|^p \, dy = \sum_{k=3}^{\infty} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) \, dy \leq \sum_{k=3}^{\infty} 8^{(n-\lambda)k} \int_{B(x_k, 8^{-k})} 1 \, dy = C(n) \sum_{k=3}^{\infty} 8^{-\lambda k} < \infty.$$

The lemma is proved. ■

Lemma 2. Let $k \geq 3$ be an integer, $x \in \mathbb{R}^n$, and $r > 0$. If $|x - x_k| < 2(4^{-k})$, then

$$r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) \, dy \leq C(n, \lambda).$$

Proof. We have two cases: (i) $2(8^{-k}) \leq |x - x_k| < 2(4^{-k})$ or (ii) $|x - x_k| < 2(8^{-k})$. For the case (i), we obtain

$$2(8^{-k}) \leq |x - x_k| < |x - y| + |y - x_k| < r + 8^{-k} \Longrightarrow r^{-\lambda} < 8^{\lambda k}$$

for every $y \in B(x, r) \cap B(x_k, 8^{-k})$. Hence

$$r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) \, dy = r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} 8^{(n-\lambda)k} \, dy \leq C(n) 8^{\lambda k} 8^{(n-\lambda)k} 8^{-nk} = C(n).$$

For the case (ii), it is easy to show that $B(x_k, 8^{-k}) \subseteq B\left(x, 3(8^{-k})\right)$, which implies

$$\begin{aligned} r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) \, dy &= r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} 8^{(n-\lambda)k} \, dy \leq 8^{(n-\lambda)k} \int_{B(x,r) \cap B(x_k, 8^{-k})} |x - y|^{-\lambda} \, dy \\ &\leq 8^{(n-\lambda)k} \int_{B\left(x, 3(8^{-k})\right)} |x - y|^{-\lambda} \, dy = C(n, \lambda) 8^{(n-\lambda)k} 8^{(n-\lambda)(-k)} = C(n, \lambda). \end{aligned}$$

Combining the two cases results above, the lemma is proved. ■

Using the Lemma 1 and the Lemma 2, we will show that the function f belongs to the Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$.

Lemma 3. $f \in L^{p,\lambda}(\mathbb{R}^n)$.

Proof. From Lemma 1, $f \in L^p_{loc}(\mathbb{R}^n)$. Now, let $x \in \mathbb{R}^n$ and $r > 0$. There are two cases: (i) $x \notin B\left(x_k, 2(4^{-k})\right)$ for every $k \geq 3$, or (ii) $x \in B\left(x_j, 2(4^{-k})\right)$ for some $j \geq 3$. Assume the case (i) holds. For every $y \in B(x,r) \cap B(x_k, 8^{-k})$, we have

$$2(4^{-k}) \leq |x - x_k| \leq |x-y| + |y - x_k| < r + 8^{-k} < r + 4^{-k}.$$

This means $r^{-\lambda} \leq 4^{\lambda k}$ and

$$r^{-\lambda}\int_{B(x,r)} |f(y)|^p\,dy = \sum_{k=3}^{\infty} r^{-\lambda} 8^{(n-\lambda)k} \int_{B(x,r)\cap B(x_k,8^{-k})} 1\,dy$$
$$\leq C(n)\sum_{k=3}^{\infty} 4^{\lambda k} 8^{(n-\lambda)k} 8^{nk} = C(n)\sum_{k=3}^{\infty} 2^{-\lambda k} < \infty. \qquad (2)$$

Assume the case (ii) holds. Then there is only one $j \geq 3$ such that $x \in B\left(x_j, 2(4^{-j})\right)$ and since $\left\{B\left(x_k, 2(4^{-k})\right)\right\}_{k\geq 3}$ is a disjoint collection. We also obtain $x \notin B\left(x_k, 2(4^{-k})\right)$ for every $k \geq 3$ with $k \neq j$. By virtue to Lemma 2 and the calculation of (2), we conclude

$$r^{-\lambda}\int_{B(x,r)} |f(y)|^p\,dy = r^{-\lambda}\sum_{k=3}^{\infty}\int_{B(x,r)\cap B(x_k,8^{-k})} f_k(y)\,dy$$

$$= r^{-\lambda}\int_{B(x,r)\cap B(x_j,8^{-j})} f_j(y)\,dy + r^{-\lambda}\sum_{\substack{k=3\\k\neq j}}^{\infty}\int_{B(x,r)\cap B(x_k,8^{-k})} f_k(y)\,dy$$

$$\leq C(n,\lambda) + C(n)\sum_{\substack{k=3\\k\neq j}}^{\infty} 2^{-\lambda k} < \infty. \qquad (3)$$

Therefore,

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0}\left(r^{-\lambda}\int_{B(x,r)} |f(y)|^p\,dy\right)^{\frac{1}{p}} \leq C(n,\lambda,p) < \infty,$$

in view of (2) and (3). This completes the proof. ∎

Lemma 4. $f \notin VL^{p,\lambda}(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$ and $0 < r < 1$. Choose an integer k such that $8^{-k} < r$. Then

$$\left(\mathcal{M}_f(r)\right)^p \geq r^{-\lambda}\int_{B(x_k,r)} |f(y)|^p\,dy \geq \int_{B(x_k,8^{-k})} f_k(y)\,dy \geq \int_{B(x_k,8^{-k})} 8^{(n-\lambda)k}\,dy$$
$$\geq C(n) 8^{nk} 8^{-nk} = C(n) > 0.$$

Hence $\mathcal{M}_f(r)$ bounded away from zero as r tends to zero. ∎

Proof of The Theorem 1. Let f be defined by (1). According to the Lemma 3 and the Lemma 4, we have $f \in L^{p,\lambda}(\mathbb{R}^n) \backslash VL^{p,\lambda}(\mathbb{R}^n)$. Hence $VL^{p,\lambda}(\mathbb{R}^n)$ is a proper subset of $L^{p,\lambda}(\mathbb{R}^n)$. ∎

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