April 2022 | Vol. 3 | No. 1 Halaman 43–48
p-ISSN: 2723-0325 e-ISSN: 2723-0333

The Total Irregularity Strength of m Copies of the Friendship Graph

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Manuscript submitted : Maret 2022;
Accepted for publication : April 2022.
doi : https://doi.org/10.30598/tensorvol3iss1pp43-48

Abstract: This paper deals with the totally irregular total labeling of the disjoin union of friendship graphs. The results shows that the disjoin union of m copies of the friendship graph is a totally irregular total graph with the exact values of the total irregularity strength equals to its edge irregular total strength.

2010 Mathematical Subject Classification: 05C78.
Key words: disjoin union, friendship graph, total irregularity strength, totally irregular total labeling

1. Introduction

In [1], Baca, *et al*. introduced a *vertex irregular total* k*- labeling* and an *edge irregular total* k*-labeling* of a graph as a modification of the well-known *irregular assignment* given by Chartrand, *et al*. [2].

A vertex irregular total k-labeling of a graph G is a function that map all the vertices and edges of G to k positive integer $\lambda: V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ such that all the vertex weights are distinct, where $w(v) = \lambda(v) + \sum_{vx \in E(G)} \lambda(vx)$. The minimum value k for which G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, denoted by $tvs(G)$. The boundaries of $tvs(G)$ is given in [1] as follow.

Theorem 1. [1] Let G be a graph (p, q) with minimum degree δ and maximum degree Δ. Then

$$\left\lceil \frac{p+r}{r+1} \right\rceil \leq tvs(G) \leq p - r + 1 \quad (1)$$

An edge irregular total k-labeling of a graph G is a function that map all the vertices and edges of G to k positive integer $\lambda: V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ such that all the edge weights are distinct, where $w(vx) = \lambda(v) + \lambda(vx) + \lambda(x)$. The minimum value k for which G has an edge irregular total k-labeling is called the

total edge irregularity strength of G, denoted by $tes(G)$. The lower bound of $tes(G)$ is given in [1] as follow.
Theorem 2. [1] Let $G = (V, E)$ be any graph with maximum degree Δ. Then

$$tes(G) \geq \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\} \quad (2)$$

Later, many results given for improving the boundaries and the exact values of $tvs(G)$ and $tes(G)$ of certain types of graphs are determined instead of finding the exact values of any graph. In [1], Baca, *et al*. have determined the $tvs(G)$ of complete graph, cycle graph, star graph, and prism graph. Nurdin, *et al* [3], Simanjuntak, *et al.* [4], and Susilawati, *et al.* [5], focused on the boundaries and exact values of the total vertex irregularity strength of trees. For the disjoint union of friendship graph, Ahmad, *et al.* [6] have showed that its total vertex irregularity strength equals to the lower bound given in equation (1) and provided corollary for m copies of friendship graph as follow.

Corolarry 3. [6] Let F_n be a friendship graph with n triangles, $n \geq 3$, and let mF_n be the disjoint union of m copies of F_n, $m \geq 2$. Then

$$tvs(mF_n) = \left\lceil\frac{2(mn+1)}{3}\right\rceil. \quad (3)$$

Many results of the total edge irregularity strength of some certain class of graphs also be provided by several authors, namely Baca, *et al*. for path graph, cycle graph, star graph, wheel graph, and friendship graph in [1], Ivanco and Jendrol [7] for trees, and many others. In [6], Ahmad, *et al*, also provided the total edge irregularity strength of the disjoint union of friendship graph, and the following corollary, which is equal to the lower bound in equation (2).

Corolarry 4. [6] Let F_n be a friendship graph with n triangles, $n \geq 3$, and let mF_n be the disjoint union of m copies of F_n, $m \geq 2$. Then

$$tes(mF_n) = mn + 1 \quad (4)$$

Motivated by both labeling, Marzuki, *et al.* [8] introduced a *totally irregular total k-labeling* of a graph G, as a total k-labeling such that for every two distinct vertices v and x, their weights $w(v)$ and $w(x)$ are distinct, and for every two distinct edges v_1v_2 and x_1x_2, their weights $w(v_1v_2)$ and $w(x_1x_2)$ are distinct. The minimum k for which G has a totally irregular total k-labeling is called the total irregularity strength of G, denoted $ts(G)$.
They [8] provided the lower bound of $ts(G)$ as follow.

Observation 5. [8] For every graph G,

$$ts(G) \geq \max\{tes(G), tvs(G)\} \quad (5)$$

They [8] proved that the lower bound is sharp for path graph of order $n \neq 2, 5$ and cycle graphs. Ramdani and Salman [9], Ramdani, *et al* [10], Tilukay, *et al.* [11-13], also confirmed the sharpness of the lower bound of the total irregularity of several types of graphs, including friendship graph F_n. More results can be seen in a survey provided by Galian in [14].

In this paper, we investigate the total irregularity strength of the disjoint union of m copies of friendship graph.

2. The Total Irregularity Strength of m Copies of the Friendship Graph

The friendship graph F_n is a set of n copies of a triangle having a common vertex as a center and the other mutually disjoint vertices.

Theorem 6. Let mF_n be the disjoint union of m copies of a friendship graph F_n, where $m \geq 2$ and $n \geq 2$. Then

$$ts(mF_n) = mn + 1.$$

Proof. Since the friendship graph F_n has $2n + 1$ vertices and $3n$ edges, the disjoint union of m copies of the friendship graph F_n is a graph of order $2mn + m$, size $3mn$, and maximum degree $2n$. Follow from equation (3-5), we obtain $ts(mF_n) \geq mn + 1$, for $m, n \geq 2$. Next, to conclude that it is the exact value of $ts(mF_n)$, we need to prove that there is a totally irregular total $(mn + 1)$ −labeling of mF_n. Let $V(mF_n) = \{v_i | 1 \leq i \leq m\} \cup \left\{x_{i_1}^j, y_{i_1}^j | 1 \leq i \leq m, 1 \leq j \leq r\right\} \cup \{x_i^{r+1}, y_i^{r+1} | 1 \leq i \leq m\} \cup \left\{x_{i_2}^j, y_{i_2}^j | 1 \leq i \leq m, 1 \leq j \leq n - r - 1\right\}$ and $E(mF_n) = \left\{v_i x_{i_1}^j, v_i y_{i_1}^j, x_{i_1}^j y_{i_1}^j | 1 \leq i \leq m, 1 \leq j \leq r\right\} \cup \{v_i x_i^{r+1}, v_i y_i^{r+1}, x_i^{r+1} y_i^{r+1} | 1 \leq i \leq m\} \cup \left\{v_i x_{i_2}^j, v_i y_{i_2}^j, x_{i_2}^j y_{i_2}^j | 1 \leq i \leq m, 1 \leq j \leq n - r - 1\right\}$. Let $t_i = ni + 1$ and $r = \left\lfloor \frac{n-1}{2} \right\rfloor$. Consider that mF_n consists of m component of form F_n. We partition n triangles of each component of form F_n into 3 parts, as follow:

i. r first triangles $v_i x_{i_1}^j y_{i_1}^j v_i$, where $1 \leq i \leq m$ and $1 \leq j \leq r$;

ii. Triangle $v_i x_i^{r+1} y_i^{r+1} v_i$, where $1 \leq i \leq m$;

iii. $n - r - 1$ triangles $v_i x_{i_2}^j y_{i_2}^j v_i$, where $1 \leq i \leq m$ and $1 \leq j \leq n - r - 1$.

Next, we construct a total labeling $\lambda: V \cup E \rightarrow \{1, 2, 3, \cdots, mn + 1\}$ defined by

$\lambda(v_i) = n(i-1) + r + 1,$ $\quad 1 \leq i \leq m;$

$\lambda\left(x_{i_1}^j\right) = n(i-1) + 1,$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq r;$

$\lambda\left(y_{i_1}^j\right) = n(i-1) + 1,$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq r;$

$\lambda\left(x_i^{r+1}\right) = n(i-1) + r + 1,$ $\quad 1 \leq i \leq m;$

$\lambda\left(y_i^{r+1}\right) = n(i-1) + r + 1,$ $\quad 1 \leq i \leq m;$

$\lambda\left(x_{i_2}^j\right) = ni + 1,$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq n - r - 1;$

$\lambda\left(y_{i_2}^j\right) = ni + 1,$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq n - r - 1;$

$\lambda\left(x_{i_1}^j y_{i_1}^j\right) = n(i-1) + j$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq r;$

$\lambda\left(v_i x_{i_1}^j\right) = n(i-1) + 2j - 1,$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq r;$

$\lambda\left(v_i y_{i_1}^j\right) = n(i-1) + 2j,$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq r;$

$\lambda\left(x_i^{r+1} y_i^{r+1}\right) = n(i-1) + r + 1,$ $\quad 1 \leq i \leq m;$

$\lambda\left(v_i x_i^{r+1}\right) = n(i-1) + r + 2,$ $\quad 1 \leq i \leq m;$

$\lambda\left(v_i y_i^{r+1}\right) = n(i-1) + r + 3,$ $\quad 1 \leq i \leq m;$

$\lambda\left(x_{i_2}^j y_{i_2}^j\right) = n(i-1) + r + j + 1,$ $\quad 1 \leq i \leq m,\ 1 \leq j \leq n - r - 1.$

$$\lambda\left(v_i x_{i_2}^j\right) = \begin{cases} n(i-1)+2j, & \text{even } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \\ n(i-1)+2j+1, & \text{odd } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \end{cases}$$
$$\lambda\left(v_i y_{i_2}^j\right) = \begin{cases} n(i-1)+2j+1, & \text{even } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \\ n(i-1)+2j+2, & \text{odd } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \end{cases}$$

From the construction above, one can check that the maximum label is $mn+1$, as on vertex $x_{i_2}^n$ for example. Next, we evaluate all the vertex-weights and all the edge-weights as follow.

For the edge-weights we have:

$w\left(x_{i_1}^j y_{i_1}^j\right) = 3n(i-1)+j+2, \quad 1 \le i \le m,\ 1 \le j \le r;$

$w\left(v_i x_{i_1}^j\right) = 3n(i-1)+r+2j+1, \quad 1 \le i \le m,\ 1 \le j \le r;$

$w\left(v_i y_{i_1}^j\right) = 3n(i-1)+r+2j+2, \quad 1 \le i \le m,\ 1 \le j \le r;$

$w\left(x_i^{r+1} y_i^{r+1}\right) = 3n(i-1)+3r+3, \quad 1 \le i \le m;$

$w\left(v_i x_i^{r+1}\right) = 3n(i-1)+3r+4, \quad 1 \le i \le m;$

$w\left(v_i y_i^{r+1}\right) = 3n(i-1)+3r+5, \quad 1 \le i \le m;$

$w\left(x_{i_2}^j y_{i_2}^j\right) = 3n(i-1)+2n+r+j+3, \quad 1 \le i \le m,\ 1 \le j \le n-r-1.$

$$w\left(v_i x_{i_2}^j\right) = \begin{cases} 3n(i-1)+n+r+2j+2, & \text{even } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \\ 3n(i-1)+n+r+2j+3, & \text{odd } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \end{cases}$$
$$w\left(v_i y_{i_2}^j\right) = \begin{cases} 3n(i-1)+n+r+2j+3, & \text{even } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \\ 3n(i-1)+n+r+2j+4, & \text{odd } n,\ 1 \le i \le m, 1 \le j \le n-r-1. \end{cases}$$

Investigating all these edge-weights, we obtain that there is no two vertices of the same weight. Specifically, this total labeling is very optimal such that all the edge-weights for a consecutive sequence of difference 1 from 3 to $|E(mF_n| + 2$.

For the vertex-weights we have:

$$w(v_i) = \begin{cases} (2n^2+n)(i-1)+2n^2+4r^2-4nr-n+5r+5, & \text{even } n,\ 1 \le i \le m; \\ (2n^2+n)(i-1)+2n^2+4r^2-4nr+n+3r+3, & \text{odd } n,\ 1 \le i \le m; \end{cases}$$

$w\left(x_{i_1}^j\right) = 3n(i-1)+3j, \quad 1 \le i \le m,\ 1 \le j \le r;$

$w\left(y_{i_1}^j\right) = 3n(i-1)+3j+1, \quad 1 \le i \le m,\ 1 \le j \le r;$

$w\left(x_i^{r+1}\right) = 3n(i-1)+3r+4, \quad 1 \le i \le m;$

$w\left(y_{i_1}^{r+1}\right) = 3n(i-1)+3r+5, \quad 1 \le i \le m;$

$$w\left(x_{i_2}^j\right) = \begin{cases} 3ni+r+3j+2, & \text{even } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \\ 3ni+r+3j+3, & \text{odd } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \end{cases}$$
$$w\left(y_{i_2}^j\right) = \begin{cases} 3ni+r+3j+3, & \text{even } n,\ 1 \le i \le m, 1 \le j \le n-r-1; \\ 3ni+r+3j+4, & \text{odd } n,\ 1 \le i \le m, 1 \le j \le n-r-1. \end{cases}$$

Investigating all these vertex-weights, we obtain the following conditions.

1. $w\left(x_{i_1}^j\right) < w\left(y_{i_1}^j\right) < w\left(x_i^{r+1}\right) < w\left(y_{i_1}^{r+1}\right) < w\left(x_{i_2}^j\right) < w\left(y_{i_2}^j\right)$;
2. $w(v_i) = w\left(x_{k_1}^j\right)$ or $w(v_i) = w\left(y_{k_1}^j\right)$ or $\cdots$ or $w(v_i) = w\left(y_{k_2}^j\right)$, for some $i < k \le m$.

For instance, Condition 2 above is occurred on graph $12F_6$. Under the total labeling λ on graph $12F_6$, we found that $w(v_1) = w\left(x_{3_2}^2\right) = 49$, $w(v_2) = w\left(x_{7_2}^3\right)$, and $w(v_3) = w\left(y_{12_1}^2\right) = 205$. It means that we need to eliminate Condition 2 to have the appropriate total labeling λ.

Since the $|V(mF_n)| < |E(mF_n|$ while λ is very optimal for labeling all the edges, there are gaps in the sequence of vertex-weights $w\left(x_{i_1}^j\right) < w\left(y_{i_1}^j\right) < w\left(x_i^{r+1}\right) < w\left(y_{i_1}^{r+1}\right) < w\left(x_{i_2}^j\right) < w\left(y_{i_2}^j\right)$. It means that the weight of v_i can be changed to eliminate Condition 2. Consider the i-th friendship graph F_n of Condition 2 (where its center-weight $w(v_i)$ equals to one of $w\left(x_{i_1}^j\right), w\left(y_{i_1}^j\right), w\left(x_i^{r+1}\right), w\left(y_{i_1}^{r+1}\right), w\left(x_{i_2}^j\right), w\left(y_{i_2}^j\right)$). Since $\lambda\left(x_{i_1}^j\right) = \lambda\left(y_{i_1}^j\right), \lambda\left(x_i^{r+1}\right) = \lambda\left(y_{i_1}^{r+1}\right)$, and $\lambda\left(x_{i_2}^j\right) = \lambda\left(y_{i_2}^j\right)$, then $w(v_i)$ can be modified to have a distinct weight that fill the gap without changing the edge-weight sequence as follow.

Let $a \neq 0$ be the minimum integer for which $w(v_i) + a$ can fill the gap.

1. For even $a > 0$, choose some triangles $v_i x_i^j y_i^j$, for some j and define $\lambda^*\left(x_i^j\right) = \lambda^*\left(y_i^j\right) = \lambda\left(x_i^j\right) - 1$, $\lambda^*\left(v_i x_i^j\right) = \lambda\left(v_i x_i^j\right) + 1$, $\lambda^*\left(v_i y_i^j\right) = \lambda\left(v_i y_i^j\right) + 1$, and $\lambda^*\left(x_i^j y_i^j\right) = \lambda\left(x_i^j y_i^j\right) + 2$.
2. For even $a < 0$, choose some triangles $v_i x_i^j y_i^j$, for some j and define $\lambda^*\left(x_i^j\right) = \lambda^*\left(y_i^j\right) = \lambda\left(x_i^j\right) + 1$, $\lambda^*\left(v_i x_i^j\right) = \lambda\left(v_i x_i^j\right) - 1$, $\lambda^*\left(v_i y_i^j\right) = \lambda\left(v_i y_i^j\right) + 1$, and $\lambda^*\left(x_i^j y_i^j\right) = \lambda\left(x_i^j y_i^j\right) - 2$.
3. For odd $a > 0$, choose triangles $v_i x_i^j y_i^j$ triangles, for some j, of the last $n - r - 1$ triangles, and define $\lambda^*\left(v_i x_i^j\right) = \lambda\left(x_i^j y_i^j\right) + \lambda\left(y_i^j\right) - \lambda(v_i)$ and $\lambda^*\left(x_i^j y_i^j\right) = \lambda\left(v_i x_i^j\right) + \lambda(v_i) - \lambda\left(y_i^j\right)$.
4. For odd $a < 0$, choose triangles $v_i x_i^j y_i^j$ triangles, for some j, of the first $r + 1$ triangles, and define $\lambda^*\left(v_i x_i^j\right) = \lambda\left(x_i^j y_i^j\right) + \lambda\left(y_i^j\right) - \lambda(v_i)$ and $\lambda^*\left(x_i^j y_i^j\right) = \lambda\left(v_i x_i^j\right) + \lambda(v_i) - \lambda\left(y_i^j\right)$.
5. Set $\lambda^*(v) = \lambda(v)$ and $\lambda^*(e) = \lambda(e)$ for each of the rest of vertices and edges.

After applying the above modification on the total labeling λ to have λ^*, we can obtain that there is no two vertex of the same weight.

Thus, the total labeling λ^* above is a totally irregular total $(mn + 1)$-labeling and the exact value of the total irregularity strength of m copies of the friendship graph is $mn + 1$. ■

3. Conclusion

By Theorem 1, we have showed that m copies of the friendship graph mF_n is a totally irregular total graph and the total irregularity strength of mF_n is equal to its total edge irregularity strength.

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